

Ministry of Education and Science of Russia  
Federal state-financed educational  
institution of higher professional education  
"Gorno-Altai State university"

G.G. Mikhailichenko

**THE MATHEMATICAL BASICS  
AND RESULTS OF THE  
THEORY OF PHYSICAL STRUCTURES**

An appendix by  
A.N. Borodin

Translation by  
N.V. Faranosov

Gorno-Altai  
The Printing Department of Gorno-Altai state university  
2016

Printed by the Editorial and Printing Council of Gorno-Altaiisk  
state university order

УДК 514.1    ББК 22.3    М 69

**Г.Г. Михайличенко**

**Математические основы и результаты теории физических  
структур:**

Монография / Михайличенко Г.Г. – Горно-Алтайск: РИО ГАГУ,  
2012, - 146 с.

**ISBN 978-5-91425-079-6**

*R e v i e w e d* by: Doctor of Physical and Mathematical Sciences  
Professor Yu. S. Vladimirov (MSU); Doctor of Physical and Mathematical  
Sciences E.E. Vitiaev (out-of-staff member IM SD RAS); Candidate of  
Physical and Mathematical Sciences associate professor V.A. Kyrov (GASU).

The theory of Physical Structures (TPS) was put forward by Professor  
Yu.I. Kulakov for the sake of classifying the laws of Physics. The history of  
the development of that theory is given in his monograph [1]. A physical  
structure is a geometry of one or two sets whose metric function assigns  
a number to every pair of points. Its phenomenological symmetry, under  
Kulakov, means that for every collection of some finite number of points all  
of their reciprocal distances are functionally related. Such geometries are  
endowed with a group symmetry under Klein, which is equivalent to the  
phenomenological symmetry, and many of them have an essential physical  
interpretation. That is why they are to be defined precisely and explored as  
purely mathematical objects. In this monograph we treat the mathematical  
basics of the TPS and present the results of attempts at classification that  
have been obtained by now. We hope that the monograph will be of interest  
for research workers and teachers, senior and post graduate students, as  
well as to all those interested in algebra, geometry and theoretical physics  
who would like to use the TPS in their research projects or could want to  
contribute to the development of its mathematical apparatus.

**ISBN 978-5-91425-079-6**

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# INTRODUCTION

To illustrate phenomenological and group symmetries in ordinary geometry as well as their relation, let us first take Euclidean plane. In the Cartesian rectangular coordinate system  $(x, y)$  the squared distance  $\rho(ij)$  between any two points  $i = (x_i, y_i)$  and  $j = (x_j, y_j)$  of it is determined by the function

$$f(ij) = \rho^2(ij) = (x_i - x_j)^2 + (y_i - y_j)^2. \quad (\text{B.1})$$

We shall take four arbitrary points  $i, j, k, l$  and write six values of the metric function (B.1):  $f(ij), f(ik), f(il), f(jk), f(jl), f(kl)$  for them. It is well known that the six reciprocal distances among any four points of an Euclidean plane are functionally related, turning into zero the Cayly-Menger determinant of the fifth order:

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & f(ij) & f(ik) & f(il) \\ 1 & f(ij) & 0 & f(jk) & f(jl) \\ 1 & f(ik) & f(jk) & 0 & f(kl) \\ 1 & f(il) & f(jl) & f(kl) & 0 \end{vmatrix} = 0. \quad (\text{B.2})$$

The geometrical meaning of the relation (B.2) is in that the volume of a tetrahedron with all the apices in one plane is equal to zero. Under Yu.I. Kulakov's terms [2], the relation (B.2), that is true for any quadruple of points  $<ijkl>$ , expresses the phenomenological symmetry of Euclidean plane.

By the metric function (B.1), it is possible to find the set of motions of Euclidean plane, i.e. the set of such smooth and invertible transformations

$$x' = \lambda(x, y), \quad y' = \sigma(x, y), \quad (\text{B.3})$$

of it with respect to which the function is preserved:  $f(i'j') = f(ij)$ .

Indeed, if the transformation (B.3) is a motion, then for the functions  $\lambda$  and  $\sigma$  of it we have the functional relation

$$(\lambda(i) - \lambda(j))^2 + (\sigma(i) - \sigma(j))^2 = (x_i - x_j)^2 + (y_i - y_j)^2,$$

where, for example,  $\lambda(i) = \lambda(x_i, y_i)$ .

By way of reducing that equation to a system of functional differential relations, it is possible to find all its solutions:

$$\left. \begin{aligned} \lambda(x, y) &= ax - \varepsilon by + c, \\ \sigma(x, y) &= bx + \varepsilon ay + d, \end{aligned} \right\} \quad (\text{B.4})$$

where  $\varepsilon = \pm 1$ ;  $a^2 + b^2 = 1$ ,  $c, d$  are arbitrary constants.

The set of all the motions (B.3) with the functions (B.4) is a group that determines the group symmetry of Euclidean plane. On the other hand, that three-parameter group of transformations of the coordinate plane  $(x, y)$  defines on it, under F. Klein [3], an Euclidean geometry. In particular, the metric function  $f(ij)$  may be found by way of solving the functional equation

$$f(x'_i, y'_i, x'_j, y'_j) = f(x_i, y_i, x_j, y_j)$$

as its two-point invariant. The general solution of that equation coincides with the metric function (B.1) with an accuracy up to a scaling transformation:

$$f(ij) = \psi((x_i - x_j)^2 + (y_i - y_j)^2),$$

where  $\psi$  is an arbitrary function of one variable.

Let us check whether there exists any relation of the phenomenological and group symmetries of an arbitrary planar geometry defined by the metric function

$$f(ij) = f(x_i, y_i, x_j, y_j), \quad (\text{B.5})$$

that generalizes the expression (B.1).

A rigid figure on a plane, under any reasonable definition of the notion of motion has three degrees of freedom. Let us consider a four-point figure  $\langle ijkl \rangle$ . Every point of it is defined by two coordinates, and the figure on the whole – by eight. The six values of the function (B.5) for that figure must be dependent, because otherwise the number of its degrees of freedom will only be as few as two:  $8 - 6 = 2$ . Thus, for any quadruple  $\langle ijkl \rangle$  there must exist a functional relation

$$\Phi(f(ij), f(ik), f(il), f(jk), f(jl), f(kl)) = 0, \quad (\text{B.6})$$

that expresses the *phenomenological symmetry* of the planar geometry with the metric function (B.5).

By virtue of simple considerations the existence of a relation (B.6) implies the existence of a three-parameter group of motions:

$$\left. \begin{aligned} x' &= \lambda(x, y; a^1, a^2, a^3), \\ y' &= \sigma(x, y; a^1, a^2, a^3), \end{aligned} \right\} \quad (\text{B.7})$$

with respect to which the metric function (B.5) is a two-point invariant:  $f(i'j') = f(ij)$  or

$$f(\lambda(i), \sigma(i), \lambda(j), \sigma(j)) = f(x_i, y_i, x_j, y_j), \quad (\text{B.8})$$

where, for example,  $\lambda(i) = \lambda(x_i, y_i; a^1, a^2, a^3)$ .

The set of all the motions (B.7) defines the *group symmetry* of the planar geometry with the metric function (B.5).

We shall note that the above considerations concerning the relation between the phenomenological and group symmetries are not only applicable to Euclidean plane, but to other two-dimensional geometries (the Lobachevski plane, the Minkowski plane, the symplectic plane, the ordinary two-dimensional sphere and so on).

H. Helmholtz in his work "On the facts underlying geometry" [4] suggested that the metric function of an  $n$ -dimensional space cannot be an arbitrary one if in that space a rigid body has  $n(n+1)/2$  degrees of freedom. But then there must exist a functional relation among all the reciprocal distances for  $n+2$  points of the rigid body, because absence of such a relation would reduce by one the number of the degrees of freedom of a rigid body of  $(n+2)$ -points with the common point spacing whose motion defines uniquely the motion of the whole solid body. So, it has been natural to suppose that the phenomenological symmetry of an  $n$ -dimensional space is impossible with an arbitrary function. For  $n=1$  and  $n=2$ , it was established in the notes [5] and [6] by the author, and for  $n=3$  – in V.H. Lev's note [7].

We shall also note that the problem of classification of all planar (two-dimensional) geometries in which the position of a figure "is determined by three conditions" was formulated for the first time by J.H. Poincare in his work "Sur les hypotheses fondamentales de la geometrie" [8].

The metric function  $f(ij)$  gives a geometry of space. Indeed, through that function it is possible to find the group of motions with respect to which it is

a two-point invariant. Group symmetry also lies in the basis of the F. Klein's "Erlangen programme" of 1872 [3], under which geometry of space is a theory of invariants of some group of its transformations. On the other hand, there appears in geometry some phenomenological symmetry expressed by some functional relation among all the reciprocal distances for a certain number of points of the space. For the first time that sort of symmetry became an object of special attention in the works by Yu.I. Kulakov [2], who made it the basic principle of his theory of physical structures [1].

Let us consider the set of states of some thermodynamic system. We shall assign to each pair of states  $\langle ij \rangle$  two numbers equal to two quantities of heat  $Q^{TS}(ij)$  and  $Q^{ST}(ij)$  which the system gives away to other bodies in the course of the transition from the state  $i$  to the state  $j$  along two different ways,  $TS$  and  $ST$ , that consist of equilibrium processes, an isothermic one ( $T = \text{const}$ ) and a adiabatic ( $S = \text{const}$ ) one:

$$\left. \begin{aligned} Q^{TS}(ij) &= (S_i - S_j)T_i, \\ Q^{ST}(ij) &= (S_i - S_j)T_j, \end{aligned} \right\} \quad (\text{B.9})$$

where  $S$  is the entropy and  $T$  is the temperature of the system.

A two-component thermal function  $Q = (Q^{TS}, Q^{ST})$  with the expressions (B.9) for its components gives on the plane  $(S, T)$  a geometry that, like the Euclidean geometry on a plane, is phenomenologically symmetric, on the one hand, and is endowed with a group symmetry, on the other.

Let us take three arbitrary states,  $i$ ,  $j$ , and  $k$ . Then, in addition to two quantities of heat, determined by the expressions (B.9), we can write four more:  $Q^{TS}(ik)$ ,  $Q^{ST}(ik)$  and  $Q^{TS}(jk)$ ,  $Q^{ST}(jk)$  for the pairs of states  $\langle ik \rangle$  and  $\langle jk \rangle$ . We may exclude of these six the three entropies,  $S_i$ ,  $S_j$ ,  $S_k$ , and the three temperatures,  $T_i$ ,  $T_j$ ,  $T_k$ , of the states  $i$ ,  $j$ ,  $k$ , which will yield as result two independent functional relations among all the quantities of

heat, determined by the following equations:

$$\left. \begin{aligned} & \left| \begin{array}{ccc} 0 & -Q^{ST}(ij) & -Q^{ST}(ik) \\ Q^{TS}(ij) & 0 & -Q^{ST}(jk) \\ Q^{TS}(ik) & Q^{TS}(jk) & 0 \end{array} \right| = 0, \\ & \left| \begin{array}{ccc} Q^{TS}(ij) & Q^{TS}(jk) & -Q^{ST}(ik) \\ Q^{TS}(ik) & 0 & -Q^{ST}(ik) \\ Q^{TS}(ik) & -Q^{ST}(ij) & -Q^{ST}(jk) \end{array} \right| = 0. \end{aligned} \right\} \quad (\text{B.10})$$

The relations (B.10), true to any triple of states  $\langle ijk \rangle$ , express the phenomenological symmetry of the dimetric geometry defined on the plane  $(S, T)$  by the two-component thermal function (B.9). The group of motions in that geometry consists of all those smooth and invertible transformations

$$S' = \lambda(S, T), \quad T' = \sigma(S, T) \quad (\text{B.11})$$

of the plane  $(S, T)$  that preserve the both components of the function (B.9):

$$\left. \begin{aligned} & (\lambda(i) - \lambda(j))\sigma(i) = (S_i - S_j)T_i, \\ & (\lambda(i) - \lambda(j))\sigma(j) = (S_i - S_j)T_j. \end{aligned} \right\} \quad (\text{B.12})$$

The solutions of that system of functional equations are readily found by the method of separating of variables, the coordinates of the states  $i$  and  $j$ :

$$\lambda(S, T) = aS + b, \quad \sigma(S, T) = T/a, \quad (\text{B.13})$$

where  $a \neq 0$ ,  $b$  are arbitrary constants.

The set of transformations (B.11) with the functions (B.13) is the group of all the motions that determine the group symmetry of the two-dimensional dimetric geometry defined on the plane  $(S, T)$  by the function (B.9). Thus, the "Erlangen programme" of F. Klein is applicable to the plane of thermodynamic states [3]. In particular, the thermal function  $Q(ij)$  may be found by way of solving the functional equation

$$Q(S'_i, T'_i, S'_j, T'_j) = Q(S_i, T_i, S_j, T_j)$$

as a two-point invariant of the set of transformations (B.11), that coincides with it with an accuracy up to the scaling transformation

$$Q(ij) = \psi((S_i - S_j)T_i, (S_i - S_j)T_j),$$



where  $\psi = (\psi^1, \psi^2)$  is a two-component function of two variables.

It is easy to establish that the phenomenological symmetry of the geometry defined on the plane  $(x, y)$  by some two-component function

$$f(ij) = f(x_i, y_i, x_j, y_j), \quad (\text{B.14})$$

where  $f = (f^1, f^2)$ , is expressed by the relation

$$\Phi(f(ij), f(ik), f(jk)) = 0, \quad (\text{B.15})$$

where  $\Phi = (\Phi_1, \Phi_2)$ . As to the group symmetry of that geometry, it is defined by the group of all the motions:

$$x' = \lambda(x, y; a^1, a^2), \quad y' = \sigma(x, y; a^1, a^2), \quad (\text{B.16})$$

that depends on the two continuous parameters  $a^1, a^2$ , with respect to which group both components of the metric function (B.14) are preserved:  $f(i'j') = f(ij)$ , being its two-point invariants. We shall note that here, just as in case of the Euclidean plane, the group symmetry is also equivalent to the phenomenological one.

Yu.I. Kulakov, in his research of physics basics [9] suggested a mathematical model of the structure of a physical law considered as a phenomenologically symmetric relation among the quantities measured in the experiment. The model, he called a *physical structure*, can be applied to ordinary geometry too, and is a peculiar geometry of two sets with a metric function assigning a number to a pair of points, belonging however not to one and the same but to two different sets. In the new geometry, naturally, a motion is introduced, as a pair of transformations of the original sets such that preserve the metric function. The totality of all the motions is a group and it determines the group symmetry of the geometry in question.

Let us consider, according the principles expounded in the note [9], Newton's 2nd law of mechanics and Ohm's law of electrodynamics, writing them in such a form that would enable us to reveal their phenomenological symmetry.

Let  $i$  be a body the mass of which is equal to  $m_i$ , and  $\alpha$  an accelerator characterized by force  $F_\alpha$ . An accelerator means some other body that, by interacting with the given one, changes its speed. The quantity measured

by experiment is the acceleration  $a_{i\alpha}$ , that the accelerator  $\alpha$  imparts to the body  $i$ . In its traditional form, Newton's well-renowned second law reads that the product of the weight of the body and the acceleration it is imparted is equal to the force applied:

$$m_i a_{i\alpha} = F_\alpha. \quad (\text{B.17})$$

We shall take two arbitrary bodies  $i$ , and  $j$  and two arbitrary accelerators  $\alpha$ , and  $\beta$ . In addition to the relation (B.17), we shall write three more:

$$m_i a_{i\beta} = F_\beta, \quad m_j a_{j\alpha} = F_\alpha, \quad m_j a_{j\beta} = F_\beta.$$

From the four relations, it is possible to eliminate the masses  $m_i$  and  $m_j$  of the bodies  $i$  and  $j$ , and the forces  $F_\alpha$  and  $F_\beta$  of the accelerators  $\alpha$  and  $\beta$ , which yields a functional relation among the accelerations only, that is defined by the equations as follows:

$$a_{i\alpha} a_{j\beta} - a_{i\beta} a_{j\alpha} = 0. \quad (\text{B.18})$$

Under Yu.I. Kulakov's terms [9] the functional relation (B.18), existing for any two bodies  $i$  and  $j$  and any two accelerators  $\alpha$  and  $\beta$ , is the phenomenologically symmetric form of Newton's second law.

Let us now look at electrodynamics. To a conductor  $i$  with resistance  $R_i$  and a source of current  $\alpha$  with an electromotive force  $\mathcal{E}_\alpha$  and the internal resistance  $r_\alpha$  we shall assign a current  $I_{i\alpha}$ , measured by an ammeter in a closed circuit:

$$I_{i\alpha} = \frac{\mathcal{E}_\alpha}{R_i + r_\alpha}. \quad (\text{B.19})$$

We shall take three arbitrary conductors  $i, j$ , and  $k$  and two arbitrary current sources  $\alpha$  and  $\beta$ . Then, in addition to the current  $I_{i\alpha}$ , under the expression (B.19) it is possible to write five more values of it:

$$I_{i\beta}, I_{j\alpha}, I_{j\beta}, I_{k\alpha}, I_{k\beta}.$$

Now, from the six expressions for the current, we can eliminate the resistances  $R_i, R_j, R_k$  of the conductors  $i, j, k$ , the electromotive forces  $\mathcal{E}_\alpha, \mathcal{E}_\beta$  and internal resistances  $r_\alpha$  and  $r_\beta$  of the current sources  $\alpha$  and  $\beta$ , which yields

the functional relation among the currents defined by the following equation:

$$\begin{vmatrix} I_{i\alpha} & I_{i\beta} & I_{i\alpha}I_{i\beta} \\ I_{j\alpha} & I_{j\beta} & I_{j\alpha}I_{j\beta} \\ I_{k\alpha} & I_{k\beta} & I_{k\alpha}I_{k\beta} \end{vmatrix} = 0. \quad (\text{B.20})$$

The functional relation (B.20), true to any three conductors  $i, j, k$  and any two sources of current  $\alpha, \beta$ , is, under Kulakov terminology [9], Ohm's law in the phenomenologically symmetric form.

Concerning the equations (B.18), (B.20) and (B.2), (B.10) that define the phenomenologically symmetric functional relations among the magnitudes measured by experiment, what attracts attention is their principal generality: for any two material bodies  $i, j$  and any two accelerators  $\alpha, \beta$  the four values of acceleration  $a$  are tied by the equation (B.18); for any three conductors  $i, j, k$  and any two current sources  $\alpha, \beta$  the six values of current  $I$  are connected by the equation (B.20); for any four points  $i, j, k, l$  of Euclidean plane the six values of the squared distances  $f = \rho^2$  among them are tied by the equation (B.2); for any three states  $i, j, k$  of a thermodynamic system the six quantities of heat  $Q = (Q^{TS}, Q^{ST})$  are tied by the equation (B.10).

In each of the four examples that we have considered we deal with a *function of a pair of points* that defines, in some general sense, the distance between them, i.e. we deal with a *metric function*: in the equation (B.18) the acceleration  $a_{i\alpha}$  from Newton's 2nd law (B.17) is such a distance between the body  $i$  and the accelerator  $\alpha$  that are points (elements) of physically different sets - a set of bodies and a set of accelerators; in the equation (B.20), similarly, the current  $I_{i\alpha}$  from Ohm's law (B.19) is the distance between the conductor  $i$  and the current source  $\alpha$  that are, in their turn, points (elements) of physically different sets - a set of conductors and a set of sources of current; in the equation (B.2) the metric function  $f$  assigns, according to the formula (B.1), to a pair of points  $i$  and  $j$  of an Euclidean plane a number  $f(ij)$  equal to the squared distance of the ordinary distance  $\rho(ij)$  between them, and in the equations (B.10), the thermal function  $Q$  assigns to a pair of states  $i$  and  $j$ , which are points of the corresponding plane of thermodynamic states, two quantities of heat,  $Q^{TS}(ij)$  and  $Q^{ST}(ij)$ , defined by the expressions (B.9), and it is natural to

consider those expressions as two distances among them.

According to Yu.I. Kulakov's definition in [9] the function of acceleration (B.17) on a set of bodies and a set of accelerators gives a *physical structure of rank (2,2)*, and the function of current (B.19) on a set of conductors and a set of current sources gives a *physical structure of rank (3,2)*. These physical structures are essentially some peculiar geometries of two sets whose phenomenological symmetry is expressed by the equations (B.18) and (B.20) respectively. Similarly, the metric function (B.1) gives on a plane a *physical structure of rank 4*, i.e. a geometry of an ordinary Euclidean plane whose phenomenological symmetry is expressed by the equation (B.2). And at last, the thermal function (B.9) gives on a plane of thermodynamical states a *physical structure of rank 3* as a two-dimensional dimetric geometry whose phenomenological symmetry is expressed by the equations (B.10).

The examples of an Euclidean plane and of a plane of thermodynamic states, defined by the metric functions (B.1) and (B.9), demonstrate that their group symmetry, defined by the set of all the motions, and their phenomenological symmetry are equivalent. It is natural to suppose that a similar situation takes place in a geometry of two sets - physical structure. Under the term 'motion' in that geometry we shall understand a unity of two simultaneous transformations of each set preserving the generalized distance between the points of any pair for which it has been determined.

The transformations

$$m' = \lambda(m), \quad F' = \sigma(F) \quad (B.21)$$

of the set of bodies and the set of accelerators comprise a motion if they preserve the function of acceleration  $a = F/m$ , determined by Newton's 2nd law (B.17):

$$\frac{\sigma(F)}{\lambda(m)} = \frac{F}{m}.$$

Concerning the functions  $\lambda$  and  $\sigma$ , a simple functional equation has been obtained that is solved by the method of separating of variables:

$$\lambda(m) = cm, \quad \sigma(F) = cF, \quad (B.22)$$

where  $c \neq 0$  is an arbitrary constant. The set of all the transformations (B.21) with the functions (B.22) is a one-parameter group of motions that

defines the group symmetry of the physical structure of rank (2,2) as of a phenomenologically symmetric geometry given on a set of bodies and a set of accelerators by a function of acceleration.

If the group of transformations (B.21) is known the function of acceleration  $a = a(m, F)$  may be found by way of solving another functional equation

$$a(m', F') = a(m, F),$$

as its two-point invariant, that determines Newton's 2nd law (B.17) with an accuracy up to a scaling transformation:

$$a = \chi(F/m),$$

where  $\chi$  is a function of one variable. Its essential meaning is in the possibility of choosing the scale of the accelerometer - the device that measures acceleration. It is clear that the physical meaning of Newton's 2nd law and its phenomenological and group symmetries do not depend on any such choice.

Let us find the transformations of the set of conductors and the set of current sources:

$$R' = \lambda(R), \quad \mathcal{E}' = \sigma(\mathcal{E}, r), \quad r' = \rho(\mathcal{E}, r), \quad (\text{B.23})$$

that preserve the function of current  $I = \mathcal{E}/(R + r)$  determined by Ohm's law (B.19):

$$\frac{\sigma(\mathcal{E}, r)}{\lambda(R) + \rho(\mathcal{E}, r)} = \frac{\mathcal{E}}{R + r}.$$

With respect to the functions  $\lambda$ ,  $\sigma$ ,  $\rho$  a functional equation has been obtained whose solution is found by the method of differentiating in independent variables  $R, \mathcal{E}, r$  and separating them:

$$\lambda(R) = aR + b, \quad \sigma(\mathcal{E}, r) = a\mathcal{E}, \quad \rho(\mathcal{E}, r) = ar - b, \quad (\text{B.24})$$

where  $a \neq 0$  and  $b$  are arbitrary constants.

The transformations (B.23) with the functions (B.24) comprise a two-parameter group of motions that defines the group symmetry of the physical structure of rank (3,2) as a phenomenologically symmetric geometry that a function of current gives on a set of conductors and a set of sources of

current. On the other hand, we may, knowing the group of transformations (B.23), by way of solving the functional equation

$$I(R', \mathcal{E}', r') = I(R, \mathcal{E}, r),$$

, find the function of current  $I = I(R, \mathcal{E}, r)$  as its two-point invariant:

$$I = \chi\left(\frac{\mathcal{E}}{R + r}\right),$$

where  $\chi$  is a function of one variable. Thus, Ohm's law (B.19) is reconstructed with an accuracy up to a scaling transformation, that does not alter the physical meaning of the law, its phenomenological or group symmetries, and so it becomes possible to choose the scale of the ammeter.

Summing up the above said we arrive at the conclusion that there exist not only physical but also mathematical prerequisites for introducing a geometry of two sets. The thing is that the metric function of such a geometry allows a group of motions that defines it uniquely. Thus, the "Erlangen programme" of 1872 of F. Klein is translated from the ordinary geometry on one set over onto the geometry of two sets, the group and phenomenological symmetries turning out to be equivalent in each.

The Appendix is written by A.N. Borodin, who has explored physical structures on abstract sets. The preliminary research has showed that such widely known algebraical structures as a heap and a group are a natural corollary of the *principle of phenomenological symmetry* - that is, of the basic postulate of the theory of physical structures.

# CHAPTER I

## Geometry as a physical structure on one set

### §1. Phenomenological and group symmetries in geometry

Let there be a set  $\mathfrak{M}$  that is an  $sn$ -dimensional manifold, where  $s$  and  $n$  are natural numbers, whose points we shall designate with Latin lower-case letters, and a function  $f : \mathfrak{S}_f \rightarrow R^s$ , where  $\mathfrak{S}_f \subseteq \mathfrak{M} \times \mathfrak{M}$ , that assigns to each pair  $\langle ij \rangle \in \mathfrak{S}_f$  some real continuum of  $s$  real numbers  $f(ij) = (f^1(ij), \dots, f^s(ij)) \in R^s$ . We shall call the two-point function  $f = (f^1, \dots, f^s)$  a *metric* one, without demanding, however, that there should exist any positive definiteness of its  $s$  components or that the axioms of the ordinary metrics should be satisfied. We shall note that in most general case  $\mathfrak{S}_f \subseteq \mathfrak{M} \times \mathfrak{M}$ , i.e., the function  $f$  does not possibly assign to each pair from  $\mathfrak{M} \times \mathfrak{M}$   $s$  numbers, but in further discussion it will be convenient to understand in the explicit writing of  $f(ij)$  that  $\langle ij \rangle \in \mathfrak{S}_f$ . Let us designate by  $U(i)$  the neighbourhood of the point  $i \in \mathfrak{M}$ , by  $U(\langle ij \rangle)$  the neighbourhood of the pair  $\langle ij \rangle \in \mathfrak{M} \times \mathfrak{M}$ , and in the similar way the neighbourhoods of the corteges of other direct products of the set  $\mathfrak{M}$  by itself.

For some cortege  $\langle k_1 \dots k_n \rangle \in \mathfrak{M}^n$  of length  $n$ , let us introduce functions  $\bar{f}^n = \bar{f}[k_1 \dots k_n]$  and  $\bar{\bar{f}}^n = \bar{\bar{f}}[k_1 \dots k_n]$ , by assigning to the point  $i \in \mathfrak{M}$  the points  $(f(ik_1), \dots, f(ik_n)) \in R^{sn}$  and  $(f(k_1i), \dots, f(k_ni)) \in R^{sn}$  respectively, if  $\langle ik_1 \rangle, \dots, \langle ik_n \rangle \in \mathfrak{S}_f$  and  $\langle k_1i \rangle, \dots, \langle k_ni \rangle \in \mathfrak{S}_f$ . We shall note that the domains of the functions  $\bar{f}^n$  and  $\bar{\bar{f}}^n$  introduced may not coincide with each other or with the set  $\mathfrak{M}$  itself.

We shall suppose in respect of the space  $\mathfrak{M}$  with  $s$ -component metric

function  $f = (f^1, \dots, f^s)$  that three axioms hold as follows:

- I.** The domain  $\mathfrak{S}_f$  of the function  $f$  is a set open and dense in  $\mathfrak{M} \times \mathfrak{M}$ .
- II.** The function  $f$  is sufficiently smooth in its domain.
- III.** In  $\mathfrak{M}^n$  a set is dense of such corteges of length  $n$  for which the rank of the function  $\bar{f}^n(\bar{f}^n)$  is maximum, and is equal to  $sn$ , in the points of the set dense in  $\mathfrak{M}$ .

Sufficient smoothness means that in the domain of the function  $f$  smooth are both the function  $f$  and all the derivatives of it of sufficiently high order. We shall call the smooth metric function  $f = (f^1, \dots, f^s)$  for which the axiom III is satisfied *nondegenerate*. We shall note that the constraint in the axioms I, II, and III by open and dense sets is introduced because the original sets may comprise exceptional subsets of smaller dimension where these axioms may fail to hold.

Suppose, further, that  $m = n + 2$ . On the basis of the original metric function  $f$ , we shall construct a function  $F$ , by assigning to the cortege  $\langle ijk \dots vw \rangle$  of length  $m$  from  $\mathfrak{M}^m$  the point  $(f(ij), f(ik), \dots, f(vw)) \in R^{sm(m-1)/2}$  whose coordinates in  $R^{sm(m-1)/2}$  are determined by the sequence of  $sm(m-1)/2$  distances for the pairs of points  $\langle ij \rangle, \langle ik \rangle, \dots, \langle vw \rangle$  of the original cortege ordered by that cortege, if these pairs all belong to  $\mathfrak{S}_f$ . We shall denote the domain of the function  $F$  by  $\mathfrak{S}_F$ . The domain  $\mathfrak{S}_F$  is obviously a set open and dense in  $\mathfrak{M}^m$ .

**Definition 1.** We shall say that the function  $f = (f^1, \dots, f^s)$  gives on an  $sn$ -dimensional manifold  $\mathfrak{M}$  a *phenomenologically symmetric geometry* (physical structure) of rank  $m = n + 2$ , if, in addition to the axioms I, II, and III, the axiom is satisfied as follows:

- IV.** There exists a set dense in  $\mathfrak{S}_F$  for each cortege  $\langle ijk \dots vw \rangle$  of length  $m = n + 2$  of which and some neighbourhood  $U(\langle ijk \dots vw \rangle)$  of it if there exists a sufficiently smooth function  $\Phi : \mathcal{E} \rightarrow R^s$  defined in some region  $\mathcal{E} \subset R^{sm(m-1)/2}$  that comprises the point  $F(\langle ijk \dots vw \rangle)$  such that  $\text{rang } \Phi = s$  in it and the set  $F(U(\langle ijk \dots vw \rangle))$  is a subset of the set of zeros of the function  $\Phi$ , i.e.

$$\Phi(f(ij), f(ik), \dots, f(vw)) = 0 \quad (1.1)$$

for all the corteges of  $U(\langle ijk \dots vw \rangle)$ .



The Axiom IV gives the essence of the principle of phenomenological symmetry. That axiom expresses the requirement of  $sm(m-1)/2$  distances among the points of any cortege of length  $m = n+2$  from  $U(<ijk\dots vw>)$  being functionally related, satisfying a system of  $s$  equations (1.1). The condition  $rang \Phi = s$  means that the equations  $\Phi = 0$  (i.e.  $\Phi_1 = 0, \dots, \Phi_s = 0$ ) are independent.

If  $x = (x^1, \dots, x^{sn})$  are local coordinates in the manifold  $\mathfrak{M}$ , then for the metric function  $f = (f^1, \dots, f^s)$  in some neighbourhood  $U(i) \times U(j)$  of each pair  $\langle ij \rangle \in \mathfrak{S}_f$  it is possible to write its local coordinate representation

$$f(ij) = f(x_i, x_j) = f(x_i^1, \dots, x_i^{s_n}, x_j^1, \dots, x_j^{s_n}), \quad (1.2)$$

whose properties are determined by the axioms II и III. Since, under the axiom III the ranks of the functions  $\bar{f}^n$  and  $\bar{\bar{f}}^n$ , equal to  $sn$ , are maximum ones, the coordinates  $x_i$  and  $x_j$  are included in the representation (1.2) in an essential way. The latter implies that no locally invertible smooth change of coordinates will result in their number in the representation being decreased, i.e. there does not exist a local system of coordinates such in which it can be written as

$$f(ij) = f(x_i^1, \dots, x_i^{n'}, x_j^1, \dots, x_j^{n''}),$$

where either  $n' < sn$  or  $n'' < sn$ . Indeed, if for example  $n' < sn$  then for any cortege  $\langle j_1 \dots j_n \rangle \in (U(j))^n$  of length  $n$  and for any point from  $U(i)$  the rank of the function  $\bar{f}^n = \bar{f}[j_1 \dots j_n]$  will a fortiori be less than  $sn$ , which is in contradiction with the axiom III. We shall note, however, that the essential dependence of the representation (1.2) on the local coordinates  $x_i$  and  $x_j$  in the general case does not guaranty that the axiom III be satisfied.

Using the expression (1.2), we shall write a local coordinate representation of the function  $F$  that we have constructed:

[illegible]

the functional matrix for the components of which

$$\left\| \begin{array}{cccccc} \frac{\partial f(ij)}{\partial x_i} & \frac{\partial f(ij)}{\partial x_j} & 0 & \dots & 0 & 0 \\ \frac{\partial f(ik)}{\partial x_i} & 0 & \frac{\partial f(ik)}{\partial x_k} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \frac{\partial f(vw)}{\partial x_v} & \frac{\partial f(vw)}{\partial x_w} \end{array} \right\| \quad (1.4)$$

has  $sm(m-1)/2$  rows and  $smn$  columns. Here, it is the functional matrix for  $s$  components of the metric function  $f = (f^1, \dots, f^s)$  in the coordinates  $x = (x^1, \dots, x^{sn})$  that is briefly designated by  $\partial f / \partial x$ :

$$\frac{\partial f}{\partial x} = \left\| \begin{array}{ccc} \frac{\partial f^1}{\partial x^1} & \dots & \frac{\partial f^1}{\partial x^{sn}} \\ \dots & \dots & \dots \\ \frac{\partial f^s}{\partial x^1} & \dots & \frac{\partial f^s}{\partial x^{sn}} \end{array} \right\|. \quad (1.5)$$

The representation (1.3) is defined by a system of  $sm(m-1)/2$  functions  $f(ij), f(ik), \dots, f(vw)$  that depend in a special manner on  $smn$  coordinates  $x_i^1, \dots, x_w^{sn}$  of all the points of the cortege  $\langle ijk \dots vw \rangle$  of length  $m = n + 2$ . Since the total number of the components of the metric function  $f$  in the system (1.3) is not bigger than the total number of the coordinates, the presence of the relation (1.1) is a nontrivial fact that will not take place in case of an arbitrary system of functions (1.3).

The function  $F$ , according to the local coordinate representation (1.3), maps the neighbourhood  $U(\langle ijk \dots vw \rangle) \subset \mathfrak{S}_F \rightarrow R^{sm(m-1)/2}$ . The matrix of that mapping is the functional matrix (1.4) of the system of functions (1.3), and its rank is the rank of the matrix.

**Theorem 1.** *For the metric function  $f = (f^1, \dots, f^s)$  to give on an  $sn$ -dimensional manifold  $\mathfrak{M}$  a phenomenologically symmetric geometry (physical structure) of rank  $m = n + 2$ , it is necessary and sufficient that the rank of the mapping  $F$  be equal to  $sm(m-1)/2 - s$  on a set dense in  $\mathfrak{S}_F$ .*

The complete proof of Theorem 1 is in the author's monograph [10] and

in his note [11].

Now we shall study the group properties of the phenomenologically symmetric geometry introduced by the definition 1 above.

Let  $U$  and  $U'$  be open regions of the manifold  $\mathfrak{M}$ , that are not necessarily connected. A smooth injective mapping

$$\lambda : U \rightarrow U' \quad (1.6)$$

is called a local *motion* if it preserves the metric function  $f = (f^1, \dots, f^s)$ . The latter means that for any pair  $\langle ij \rangle \in \mathfrak{S}_f$ , such that  $i, j \in U$ , and the corresponding pair  $\langle \lambda(i), \lambda(j) \rangle$  if it belongs to  $\mathfrak{S}_f$ , the equality

$$f(\lambda(i), \lambda(j)) = f(ij), \quad (1.7)$$

takes place that is satisfied for each of the components  $f^1, \dots, f^s$  of the metric function  $f$ .

The set of all the motions (1.6) is a local group of transformations for which the metric function, according to the equality (1.7) is a *two-point invariant*. If the metric function  $f$  is defined explicitly, in its coordinate representation (1.2) for example, then the equality (1.7) is a functional equation, the solution of which gives the complete group of local motions (1.6). All we know about the metric function is that it is nondegenerate and satisfies some system of  $s$  equations (1.1). But it turns out to be enough to establish the existence of the  $sn(n+1)/2$ -parameter group of its motions.

For the sake of making further discussion clearer, we shall reproduce in our designation the definition of the local Lie group of transformations, following the monograph "Topological Groups" by L.S. Pontriagin (see [12], P. 435). Let  $G^r$  be an  $r$ -dimensional local Lie group and  $U$  – some region of the manifold  $\mathfrak{M}$ . Suppose, each element  $a \in G^r$  is assigned a continuous in  $a$  injective mapping  $\lambda_a : U \rightarrow U'$  of the region  $U$  into some region  $U'$  of the manifold  $\mathfrak{M}$ , attributing to each point  $i \in U$  some point  $i' \in U'$ , i.e.  $i' = \lambda_a(i) = \lambda(i, a)$ . We shall say that  $G^r$  is the local Lie group of transformations of the region  $U$  if three conditions hold as follows:

1. A unit  $e$  of the group  $G^r$  is corresponded by an identical transformation  $i' = \lambda(i, e) = i$  of the region  $U$  on itself and  $\lambda(\lambda(i, a), b) = \lambda(i, ab)$ , i.e. the

product  $ab \in G^r$  is corresponded by a composition of transformations: first  $\lambda_a$  and then  $\lambda_b$  (another order is possible:  $\lambda(\lambda(i, a), b) = \lambda(i, ba)$ ).

2. The transformation  $\lambda_a$  is only an identity on condition that  $a$  is a unit  $e$  of the group  $G^r$ .

3. In the coordinate form  $\lambda(i, a)$  there is a sufficiently differentiable function of the point  $i \in U$  and the element  $a \in G^r$ .

The group of transformations that we have just defined is, under condition 2, effective and so the elements of the group  $G^r$  may themselves be considered transformations. That is, it is possible to speak of an  $r$ -dimensional local group of transformations of the manifold  $\mathfrak{M}$  that we shall designate by  $G^r(\lambda)$ . Thus, there is an effective smooth action of the group  $G^r$  defined in the region  $U$ , the conditions 1, 2, and 3 being satisfied for some part of it, i.e. for some, depending on  $U$ , neighbourhood of the unit element  $e \in G^r$ .

In the further discussion it will be convenient to consider that the region  $U \subset \mathfrak{M}$  is not necessarily connected, and may for example consist of two connected regions:  $U = U_1 \cup U_2$ ,  $U_1 \cap U_2$  being equal to  $\emptyset$ .

**Definition 2.** We shall say that the function  $f = (f^1, \dots, f^s)$  gives on an  $sn$ -dimensional manifold  $\mathfrak{M}$  a *geometry endowed with a group symmetry* of degree  $sn(n+1)/2$ , if, in addition to the axioms I, II, and III, one more axiom holds as follows:

**IV'.** There exists a set open and dense in  $\mathfrak{M}$  for each point  $i$  of which there is an effective smooth action of an  $sn(n+1)/2$ -dimensional local Lie group defined in some neighbourhood  $U(i)$ , such that its actions in the neighbourhoods  $U(i)$ ,  $U(j)$  of two points  $i, j$  coincide in the intersection  $U(i) \cap U(j)$  and the function  $f(ij)$ , with respect to each of its  $s$  components is a two-point invariant of the corresponding group of transformations of the neighbourhood  $U(i) \times U(j)$ .

The group of transformations mentioned in the axiom IV' defines the local mobility of rigid figures in an  $sn$ -dimensional space  $\mathfrak{M}$ , similar to the mobility of solid bodies in the Euclidean space. We shall note that that does not imply global mobility, for though the local actions of the group  $G^{sn(n+1)/2}$  are defined, according to axiom IV', in some neighbourhood of each point of a set open and dense in  $\mathfrak{M}$ , it may appear that there is only one single element of the group acting in the whole set. The set of

pairs  $\langle ij \rangle$  for which the metric function  $f$  is defined and is the two-point invariant of the group of local transformations of the manifold  $\mathfrak{M}$  is, obviously, open and dense in  $\mathfrak{M} \times \mathfrak{M}$ . We shall also say that the metric function *allows* an  $sn(n+1)/2$ -dimensional local Lie group of local motions. It also follows from the axiom  $IV'$  that on a set open and dense in  $\mathfrak{M}$  there is an  $sn(n+1)/2$ -dimensional linear family of smooth vector fields  $X$  defined that is commutation closed, i.e. an algebra of Lie transformations (see [12], §60). In some local systems of coordinates, let us write basic vector fields in operator form:

$$X_\omega = \lambda_\omega^\mu(x) \partial / \partial x^\mu, \quad (1.8)$$

where  $\omega = 1, 2, \dots, sn(n+1)/2$ , and by the dummy index  $\mu$  an operation of summation is performed within the range 1 to  $sn$ . The metric function  $f = (f^1, \dots, f^s)$  will be the two-point invariant of the local Lie group of transformations of some neighbourhoods  $U(i)$  and  $U(j)$  of the points  $i$  and  $j$  if and only if it satisfies componentwise the system of  $sn(n+1)/2$  equations

$$X_\omega(i)f(ij) + X_\omega(j)f(ij) = 0 \quad (1.9)$$

with the operators (1.8):

$$\lambda_\omega^\mu(i) \partial f(ij) / \partial x_i^\mu + \lambda_\omega^\mu(j) \partial f(ij) / \partial x_j^\mu = 0,$$

where, for example,  $\lambda_\omega^\mu(i) = \lambda_\omega^\mu(x_i) = \lambda_\omega^\mu(x_i^1, \dots, x_i^{sn})$  (see [13], Pp. 229 and 237).

**Theorem 2.** *For the function  $f = (f^1, \dots, f^s)$  to give on an  $sn$ -dimensional manifold an  $\mathfrak{M}$  geometry endowed with the group symmetry of degree  $sn(n+1)/2$  it is necessary and sufficient that the rank of the mapping  $F$  be equal to  $sm(m-1)/2 - s$ , where  $m = n+2$ , on a set dense in  $\mathfrak{S}_F$ .*

The complete proof of theorem 2, as well as of theorem 4 below, is in the author's monograph [10], and in his note [11].

The final and obvious result of the above said is the conclusion of the phenomenological and group symmetries of the geometry defined on an  $sn$ -dimensional manifold  $\mathfrak{M}$  by the function  $f = (f^1, \dots, f^s)$  being equivalent.

That equivalence is a corollary of Theorems 1 and 2 of this paragraph, the necessary and sufficient conditions of which concerning the rank of the mapping  $F$  coincide.

**Theorem 3.** *For the function  $f = (f^1, \dots, f^s)$  to give on an  $sn$ -dimensional manifold  $\mathfrak{M}$  a phenomenologically symmetric geometry (physical structure) of rank  $m = n + 2$  it is necessary and sufficient that that function give on  $\mathfrak{M}$  a geometry endowed with a group symmetry of degree  $sn(n + 1)/2$ .*

We shall note that the condition concerning the rank of the mapping  $F$  may be formulated as a fourth axiom of the definition of the geometry. Such a geometry will be phenomenologically symmetric, on the one hand, and will be endowed with a group symmetry, on the other, and both symmetries will be, in the sense of Theorem 3, equivalent.

**Theorem 4.** *The dimension of the local group of motions allowed by the metric function  $f = (f^1, \dots, f^s)$ , giving on an  $sn$ -dimensional manifold  $\mathfrak{M}$  a phenomenologically symmetric geometry of rank  $m = n + 2$ , or a geometry endowed with a group symmetry of degree  $sn(n + 1)/2$ , is not bigger than that degree.*

Thus, rigid figures and solid bodies have not more than  $sn(n + 1)/2$  degrees of freedom, in their motion in space.

## §2. Classification of one-, two- and three-dimensional geometries

In this paragraph, we shall give complete classifications of the unimetric phenomenologically symmetric geometries where a one-component metric function  $f$  with  $s = 1$  assigns to a pair of points one number. By now, such classifications have only been built for the one-dimensional, two-dimensional, and three-dimensional geometries, i.e. for  $n = 1, 2, 3$ . Using the method of classifying phenomenologically symmetric geometries that we have developed,

when applying it to geometries of higher dimensionality, we encounter serious technical difficulties that we have not yet overcome. It is possible all those difficulties are the shortcomings of the method itself, but we have not yet discovered other and more effective methods.

The coordinate representation of a metric function, while a transition from one system of coordinates to another performed, changes too. For instance, the metric function of the Euclidean plane in the Cartesian rectangular coordinate system  $(x, y)$  is defined by the expression (B.1), while in the polar coordinate system  $(r, \varphi)$  by another:

$$f(ij) = r_i^2 + r_j^2 - 2r_i r_j \cos(\varphi_i - \varphi_j).$$

An additional scaling transformation  $\psi(f) \rightarrow f$ , where  $\psi$  is an arbitrary function of one variable, will change the original coordinate representation (B.1) still more, to the point of it becoming hardly recognizable. It is most natural to choose such a system of coordinates in the manifold  $\mathfrak{M}$  and perform such scaling transformation of the metric function itself that would make the coordinate representation of it as simple as possible. That is why the theorems of classification that follow are formulated with an accuracy up to change of coordinates and a scaling transformation.

Let us first consider, following note [5] the simplest ( $s = 1, n = 1$ ) phenomenologically symmetric geometry of rank 3. Such geometry is defined on a one-dimensional manifold  $\mathfrak{M}$  by the metric function  $f : \mathfrak{S}_f \rightarrow R$ , where  $\mathfrak{S}_f \subseteq \mathfrak{M} \times \mathfrak{M}$ . Its coordinate representation is determined by the expression

$$f(ij) = f(x_i, x_j), \tag{2.1}$$

where  $x$  is a local coordinate in the manifold. The metric function (2.1) will be nondegenerate, i.e. satisfy the axiom III of §1, on condition of both derivatives in the coordinates  $x_i$  and  $x_j$  being unequal to zero for the set of pairs  $\langle ij \rangle$  dense in  $\mathfrak{M} \times \mathfrak{M}$ . The equation expressing the phenomenological symmetry of the geometry in question, under the axiom IV of §1, establishes a functional relation of the three distances  $f(ij)$ ,  $f(ik)$ , and  $f(jk)$  for the set of triples  $\langle ijk \rangle$  dense in  $\mathfrak{M}^3$ :

$$\Phi(f(ij), f(ik), f(jk)) = 0. \tag{2.2}$$

**Theorem 1.** *With an accuracy up to a scaling transformation  $\psi(f) \rightarrow f$  and in a suitably chosen system of the local coordinate  $x$  the nondegenerate metric function (2.1) that defines on a one-dimensional manifold a phenomenologically symmetric geometry of rank 3 with the relation (2.2) may be represented with the following canonical expression:*

$$f(ij) = x_i - x_j. \quad (2.3)$$

The equation (2.2) expressing the phenomenological symmetry of that geometry is readily found:  $f(ij) - f(ik) + f(jk) = 0$ . The local invertible transformation  $x' = \lambda(x)$  of the one-dimensional manifold  $\mathfrak{M}$  with the derivative  $\lambda'(x)$  unequal to zero will be a motion if it preserves the metric function (2.3):  $\lambda(x_i) - \lambda(x_j) = x_i - x_j$ . The functional equation of the set of motions is easy to solve by the method of separating of variables:  $\lambda(x) = x + a$ , where  $a$  is an arbitrary constant. The respective one-parameter group of motions  $x' = x + a$  determines the group symmetry of degree 1 of the phenomenologically symmetric geometry of rank 3 defined on a one-dimensional manifold  $\mathfrak{M}$  by the metric function (2.3). In conclusion, we shall note that the two-point invariant of the group of motions satisfies the functional equation  $f(x_i + a, x_j + a) = f(x_i, x_j)$ . That equation is solved by reducing it to a linear homogeneous differential equation in partial derivatives:  $f(ij) = \psi(x_i - x_j)$ , where  $\psi$  is an arbitrary function of one variable, wherefrom it can be seen that by the group of motions the metric function is reconstructed uniquely with an accuracy up to a scaling transformation  $\psi(f) \rightarrow f$ .

Let us now proceed to the two-dimensional ( $s = 1, n = 2$ ) phenomenologically symmetric geometries that are defined on a two-dimensional manifold  $\mathfrak{M}$  by the metric function  $f : \mathfrak{S}_f \rightarrow R$ , where  $\mathfrak{S}_f \subseteq \mathfrak{M} \times \mathfrak{M}$ . Its coordinate representation is determined by the expression

$$f(ij) = f(x_i, y_i, x_j, y_j), \quad (2.4)$$

where  $(x, y)$  are local coordinates in the manifold. If that function really gives such a geometry then, under the axiom IV of §1 the six values of it



for the quadruple  $\langle ijkl \rangle$  are functionally related:

$$\Phi(f(ij), f(ik), f(il), f(jk), f(jl), f(kl)) = 0. \quad (2.5)$$

Obviously, the nondegenerate metric function (2.4) must, under the axiom III of §1, satisfy the following two conditions:

$$\left. \begin{aligned} \partial(f(ik), f(il))/\partial(x_i, y_i) &\neq 0, \\ \partial(f(kj), f(lj))/\partial(x_j, y_j) &\neq 0 \end{aligned} \right\} \quad (2.6)$$

for the open and dense in  $\mathfrak{M}^3$  set of triples  $\langle ikl \rangle$  and  $\langle klj \rangle$ .

The Euclidean plane with the metric function (B.1) and the functional relation (B.2) that expresses its phenomenological symmetry, that was discussed in the Introduction by way of an example, is such a geometry. But how many of them are there? That question is answered by the theorem as follows (see [6]):

**Theorem 2.** *With an accuracy up to a scaling transformation  $\psi(f) \rightarrow f$  and in a suitably chosen system of local coordinates  $(x, y)$  the nondegenerate metric function (2.4) defining on a two-dimensional manifold a phenomenologically symmetric geometry of rank 4 with the relation (2.5), may be represented by one of the following eleven canonical expressions:*

$$f(ij) = (x_i - x_j)^2 + (y_i - y_j)^2, \quad (2.7)$$

$$f(ij) = \sin y_i \sin y_j \cos(x_i - x_j) + \cos y_i \cos y_j, \quad (2.8)$$

$$f(ij) = \operatorname{sh} y_i \operatorname{sh} y_j \cos(x_i - x_j) - \operatorname{ch} y_i \operatorname{ch} y_j, \quad (2.9)$$

$$f(ij) = (x_i - x_j)^2 - (y_i - y_j)^2, \quad (2.10)$$

$$f(ij) = \operatorname{ch} y_i \operatorname{ch} y_j \cos(x_i - x_j) - \operatorname{sh} y_i \operatorname{sh} y_j, \quad (2.11)$$

$$f(ij) = x_i y_j - x_j y_i, \quad (2.12)$$

$$f(ij) = \frac{y_i - y_j}{x_i - x_j}, \quad (2.13)$$

$$f(ij) = ((x_i - x_j)^2 - (y_i - y_j)^2) \exp \left( 2\beta \operatorname{ar}(\operatorname{c}) \operatorname{th} \frac{y_i - y_j}{x_i - x_j} \right), \quad (2.14)$$

$$f(ij) = (x_i - x_j)^2 \exp \left( 2 \frac{y_i - y_j}{x_i - x_j} \right), \quad (2.15)$$

$$f(ij) = ((x_i - x_j)^2 + (y_i - y_j)^2) \exp \left( 2\gamma \arctg \frac{y_i - y_j}{x_i - x_j} \right), \quad (2.16)$$

$$f(ij) = \frac{(x_i - x_j)^2 + \varepsilon_i y_i^2 + \varepsilon_j y_j^2}{y_i y_j}, \quad (2.17)$$

where  $\beta > 0$  and  $\beta \neq 1$ ;  $\gamma > 0$ ;  $\varepsilon_i = 0, \pm 1$ ;  $\varepsilon_j = 0, \pm 1, \varepsilon_i$  not necessarily being equal to  $\varepsilon_j$ .

Six of the expressions, (2.7)–(2.12), define metric functions of the two-dimensional geometries that are well-known: **(2.7)** – of the *Euclidean plane*; **(2.8)** – of the *two-dimensional sphere* in the three-dimensional Euclidean space; **(2.9)** – of the *Lobachevski plane* as a two-dimensional two-sheet hyperboloid in the three-dimensional pseudo-Euclidean space; **(2.10)** – of the *Minkowski plane*; **(2.11)** – of the *two-dimensional two-sheet hyperboloid* in the three-dimensional pseudo-Euclidean space; **(2.12)** – of the *symplectic plane*.

The existence of four metric functions, (2.13)–(2.16) defining two-dimensional phenomenologically symmetric geometries of rank 4 was established for the first time by the author [6]. Professor A.V. Shirokov (the Chair of Geometry of the Kazan State University) drew the author's attention to the possibility to write three of the metric functions, (2.14), (2.15) and (2.16), uniformly, using the three types of complex numbers:

$$f(ij) = (z_i - z_j) \overline{(z_i - z_j)} \exp 2\gamma \arg(z_i - z_j),$$

where  $z = x + ey$ ,  $\bar{z} = x - ey$ ,  $e^2 = +1$ , and  $\gamma > 0$ , and, additionally,  $\gamma \neq 1$  for the expression (2.14);  $e^2 = 0$  and  $\gamma = 1$  for the expression (2.15);  $e^2 = -1$  and  $\gamma > 0$  for the expression (2.16). Thus, all the three possible types of complex numbers on the plane, namely double ( $e^2 = +1$ ), dual ( $e^2 = 0$ ), and common ( $e^2 = -1$ ), fit quite naturally into the complete classification of the two-dimensional phenomenologically symmetric geometries of rank 4. Apparently, the respective geometries have not ever got under scrutiny by the geometricians, and so have no universally accepted conventional names. The two-dimensional geometry with the metric function **(2.16)** was given by the author the name of the *Helmholtz plane*, for the circle in it is the logarithmic spiral, which was stated in a few words in his work [4]

by Helmholtz, who thought it to be a negative feature of such a geometry. Correspondingly, the metric function (2.14) defines a *pseudo-Helmholtz plane*, and the metric function (2.15) a *dual-Helmholtz plane*. The metric function (2.13) defines the so called *simplicial plane*, the name suggested to the author by the well-known geometrician R. Pimenov, in whose papers one comes across it. The last of the expressions, (2.17), defines a metric function that gives a two-dimensional phenomenologically symmetric geometry *on a disconnected two-dimensional manifold*, on the connected components of which there will be present either the symplectic plane ( $\varepsilon_i = \varepsilon_j = 0$ ) or the Lobachevski plane on the Poincare model ( $\varepsilon_i = \varepsilon_j = +1$ ), or the two-dimensional one-sheet hyperboloid ( $\varepsilon_i = \varepsilon_j = -1$ ).

The phenomenological symmetry of rank 4 expressed by the expression (2.5), for all the two-dimensional geometries mentioned in Theorem 2, is easily established by the rank of the functional matrix for the six functions,  $f(ij), f(ik), f(il), f(jk), f(jl), f(kl)$ , that depend in a special manner on the eight variables, the coordinates  $x_i, y_i, x_j, y_j, x_k, y_k, x_l, y_l$  of four points of the cortege  $\langle ijkl \rangle$ . The rank of that matrix, as is checked by computer, is equal to 5, which indicates the presence of the functional relation defined by the equation (2.5) and expressing the phenomenological symmetry of all the eleven geometries (2.7) – (2.17). As to the equation (2.5) itself, in the explicit form it has been found for all the two-dimensional geometries defined by the metric functions (2.14), (2.15), and (2.16), except for the Helmholtz geometry. For example, for the Euclidean plane – (2.7) and the pseudo-Euclidean Minkowski plane – (2.10) that will be the equation (B.2) from the Introduction, that turns into zero the Cayley-Menger determinant of the fifth order for the quadruple  $\langle ijkl \rangle$  of the points of those planes. For five other two-dimensional geometries, i.e. the two-dimensional sphere – (2.8), the Lobachevski plane – (2.9), one-sheet hyperboloid – (2.11), the symplectic plane – (2.12), and the geometry on an unconnected two-dimensional manifold – (2.17), in the equation (2.5) on the left there stands a gramian of the fourth order for the quadruple  $\langle ijkl \rangle$ , the diagonal elements of the determinant being the values of the metric function for the diagonal pairs  $\langle ii \rangle, \langle jj \rangle, \langle kk \rangle, \langle ll \rangle$ . For example, for the two-dimensional sphere – (2.8) and the one-sheet hyperboloid – (2.11), the

equation (2.5) will be as follows:

$$\begin{vmatrix} 1 & f(ij) & f(ik) & f(il) \\ f(ij) & 1 & f(jk) & f(jl) \\ f(ik) & f(jk) & 1 & f(kl) \\ f(il) & f(jl) & f(kl) & 1 \end{vmatrix} = 0.$$

There has also been found the explicit form of the equation (2.5) for the simplicial plane defined by the metric function (2.13):

$$\begin{vmatrix} f(ij) - f(jk) & f(jk) - f(ik) & 0 \\ f(ij) - f(jl) & 0 & f(il) - f(jl) \\ 0 & f(ik) - f(kl) & f(il) - f(kl) \end{vmatrix} = 0.$$

As has been said above, for the Helmholtz planes – (2.14), (2.15) and (2.16) the equation (2.15) has not been discovered. There is a supposition that it cannot be written through the known elementary functions.

The group symmetry of two-dimensional geometries is a natural corollary of the phenomenological symmetry, under Theorem 3 of §1. The locally invertible transformation

$$x' = \lambda(x, y), \quad y' = \sigma(x, y),$$

satisfying the condition of  $\partial(\lambda, \sigma)/\partial(x, y) \neq 0$ , will be a motion if it preserves the metric function (2.4):

$$f(\lambda(i), \sigma(i), \lambda(j), \sigma(j)) = f(x_i, y_i, x_j, y_j), \quad (2.18)$$

where, for example,  $\lambda(i) = \lambda(x_i, y_i)$ . By way of solving the functional equation (2.18) of each of the two-dimensional geometries (2.7) – (2.17) the complete local three-parameter group of motions can be found that defines the group symmetry of it of degree 3. The metric function (2.4) is also a solution of the differential equation

$$X(i)f(ij) + X(j)f(ij) = 0, \quad (2.19)$$

with the operators  $X = \lambda(x, y)\partial/\partial x + \sigma(x, y)\partial/\partial y$  of the corresponding three-dimensional Lie algebra. As to the equation (2.19), it can be considered a functional one for the coefficients  $\lambda$  and  $\sigma$  of the operator  $X$ . It turns out that so interpreted and with the known metric function (2.4) the equation (2.19)

can be solved by employing simpler methods (see [14]), than the original functional equation (2.18). We shall remind that under the known Lie theorems, there is between a Lie group and the corresponding Lie algebra a one-to-one correspondence.

Let us also consider the three-dimensional ( $s = 1, n = 3$ ) phenomenologically symmetric geometries defined on a three-dimensional manifold  $\mathfrak{M}$  by the metric function  $f : \mathfrak{S}_f \rightarrow R$ , where  $\mathfrak{S}_f \subseteq \mathfrak{M} \times \mathfrak{M}$ . The coordinate representation for it is determined by the expression

$$f(ij) = f(x_i, y_i, z_i, x_j, y_j, z_j), \quad (2.20)$$

where  $(x, y, z)$  are local coordinates in the manifold. If the function gives such a geometry, then, under the axiom IV of §1 the ten values of it for the quintuple  $\langle ijklm \rangle$  are functionally related:

$$\Phi(f(ij), f(ik), f(il), f(im), f(jk), f(jl), f(jm), f(kl), f(km), f(lm)) = 0. \quad (2.21)$$

The nondegenerate metric function (2.20) must, under the axiom III of §1, obviously satisfy the two conditions as follows:

$$\left. \begin{aligned} \partial(f(ik), f(il), f(im))/\partial(x_i, y_i, z_i) &\neq 0, \\ \partial(f(kj), f(lj), f(mj))/\partial(x_j, y_j, z_j) &\neq 0 \end{aligned} \right\} \quad (2.22)$$

for the open and dense in  $\mathfrak{M}^4$  set of quadruples  $\langle iklm \rangle$  and  $\langle klmj \rangle$ .

An example of a three-dimensional phenomenologically symmetric geometry is the three-dimensional Euclidean space. For the metric function  $f(ij)$  assigning to the pair of points  $\langle ij \rangle$  the squared ordinary distance, in the Cartesian coordinate system  $(x, y, z)$  the representation will be as follows:

$$f(ij) = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2.$$

It is well known that in the Euclidean space the ten reciprocal distances for the quintuple  $\langle ijklm \rangle$  of points are among themselves functionally related, turning into zero the Cayley-Menger determinant of the sixth order the structure of which is similar to that of the determinant of the fifth order

(B.2):

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & f(ij) & f(ik) & f(il) & f(im) \\ 1 & f(ij) & 0 & f(jk) & f(jl) & f(jm) \\ 1 & f(ik) & f(jk) & 0 & f(kl) & f(km) \\ 1 & f(il) & f(jl) & f(kl) & 0 & f(lm) \\ 1 & f(im) & f(jm) & f(km) & f(lm) & 0 \end{vmatrix} = 0.$$

The complete classification of the three-dimensional phenomenologically symmetric geometries of rank 5 was built by V.H. Lev. We shall reproduce it after his note [7]:

**Theorem 3.** *With an accuracy up to a scaling transformation  $\psi(f) \rightarrow f$  and in a suitably chosen system of local coordinates  $(x, y, z)$  the nondegenerate metric function (2.20) that gives on a three-dimensional manifold a phenomenologically symmetric geometry of rank 5 with the relation (2.21) may be represented by one of the following fifteen canonical expressions:*

$$f(ij) = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2, \quad (2.23)$$

$$f(ij) = \sin z_i \sin z_j [\sin y_i \sin y_j \cos(x_i - x_j) + \cos y_i \cos y_j] + \cos z_i \cos z_j, \quad (2.24)$$

$$f(ij) = \operatorname{sh} z_i \operatorname{sh} z_j [\sin y_i \sin y_j \cos(x_i - x_j) + \cos y_i \cos y_j] - \operatorname{ch} z_i \operatorname{ch} z_j, \quad (2.25)$$

$$f(ij) = (x_i - x_j)^2 + (y_i - y_j)^2 - (z_i - z_j)^2, \quad (2.26)$$

$$f(ij) = \operatorname{ch} z_i \operatorname{ch} z_j [\sin y_i \sin y_j \cos(x_i - x_j) + \cos y_i \cos y_j] - \operatorname{sh} z_i \operatorname{sh} z_j, \quad (2.27)$$

$$f(ij) = \operatorname{ch} z_i \operatorname{ch} z_j [\operatorname{ch} y_i \operatorname{ch} y_j \cos(x_i - x_j) - \operatorname{sh} y_i \operatorname{sh} y_j] - \operatorname{sh} z_i \operatorname{sh} z_j, \quad (2.28)$$

$$f(ij) = x_i y_j - x_j y_i + z_i - z_j, \quad (2.29)$$

$$f(ij) = \frac{y_i - y_j}{x_i - x_j} + z_i + z_j, \quad (2.30)$$

$$f(ij) = \frac{y_i - y_j}{x_i - x_j} \exp(z_i + z_j), \quad (2.31)$$

$$f(ij) = [(x_i - x_j)^2 - (y_i - y_j)^2] \exp 2(z_i + z_j), \quad (2.32)$$

$$f(ij) = [(x_i - x_j)^2 + (y_i - y_j)^2] \exp 2(z_i + z_j), \quad (2.33)$$

$$f(ij) = [(x_i - x_j)^2 - (y_i - y_j)^2] \exp[2(\beta \operatorname{ar}(c) \operatorname{th} \frac{y_i - y_j}{x_i - x_j} + z_i + z_j)], \quad (2.34)$$

$$f(ij) = (x_i - x_j)^2 \exp[2(\frac{y_i - y_j}{x_i - x_j} + z_i + z_j)], \quad (2.35)$$

$$f(ij) = [(x_i - x_j)^2 + (y_i - y_j)^2] \exp[2(\gamma \operatorname{arctg} \frac{y_i - y_j}{x_i - x_j} + z_i + z_j)], \quad (2.36)$$

$$f(ij) = \frac{(x_i - x_j)^2 \pm (y_i - y_j)^2 + \varepsilon_i z_i^2 + \varepsilon_j z_j^2}{z_i z_j}, \quad (2.37)$$

where  $\beta > 0$  and  $\beta \neq 1$ ;  $\gamma > 0$ ;  $\varepsilon_i = 0, \pm 1$ ;  $\varepsilon_j = 0, \pm 1, \varepsilon_i$  not necessarily equal to  $\varepsilon_j$ .

The seven expressions (2.23)–(2.29) determine the metric functions of well-known three-dimensional geometries: **(2.23)** – *the Euclidean space* as a natural three-dimensional extension of the Euclidean plane with the metric function (2.7); **(2.24)** – *the three-dimensional sphere* in the four-dimensional Euclidean space; **(2.25)** – *The Lobachevski space* as of a three-dimensional two-sheet hyperboloid in the four-dimensional pseudo-Euclidean space of the signature  $< + + + - >$ ; **(2.26)** – *the Minkowski space* as of a natural three-dimensional extension of the Minkowski plane with the metric function (2.10); **(2.27)** – *three-dimensional one-sheet hyperboloid I* in the four-dimensional pseudo-Euclidean space of the same signature  $< + + + - >$ ; **(2.28)** – *three-dimensional one-sheet hyperboloid II* in the four-dimensional

pseudo-Euclidean space, but of another signature  $< ++ -- >$ ; **(2.29)** – *the symplectic space* as a natural extension of the symplectic plane with the metric function (2.12) for the odd dimensionality equal to three.

The next seven expressions, (2.30)–(2.36), define the metric functions for the three-dimensional geometries discovered by V.H.Lev [15], which have never been explored, and so have no well-established and unanimously accepted names: **(2.30)** – *simplicial space I* as the additive three-dimensional phenomenologically symmetric extension of the simplicial plane with the metric function (2.13); **(2.31)** – *simplicial space II* as the multiplicative three-dimensional phenomenologically symmetric extension of the simplicial plane with the metric function (2.13); **(2.32)** – *the special phenomenologically symmetric extension of the Minkowski plane* with the metric function (2.10); **(2.33)** – *the special three-dimensional phenomenologically symmetric extension of the Euclidean plane* with the metric function (2.7); **(2.34)** – *pseudo-Helmholtz space* as the three-dimensional phenomenologically symmetric extension of the pseudo-Helmholtz plane with the metric function (2.14); **(2.35)** – *dual-Helmholtz space* as the three-dimensional phenomenologically symmetric extension of the dual-Helmholtz plane with the metric function (2.15); **(2.36)** – *Helmholtz space* as the three-dimensional phenomenologically symmetric extension of the Helmholtz plane with the metric function (2.16).

And the last one, **(2.37)**, defines the metric function of the three-dimensional geometry *on an unconnected three-dimensional manifold*, on the connected components of which it gives either the extensions (2.32), (2.33), or the spheres (2.25), (2.27), (2.28) in four-dimensional pseudo-Euclidean spaces.

The phenomenological symmetry of each of the fifteen above said three-dimensional geometries given by the metric functions (2.23)–(2.37) is proved by the rank of the respective functional matrix for the ten functions  $f(ij), f(ik), f(il), f(im), f(jk), f(jl), f(jm), f(kl), f(km), f(lm)$ , that depend in a special manner on the fifteen variables – the coordinates  $x_i, y_i, z_i, x_j, y_j, z_j, x_k, y_k, z_k, x_l, y_l, z_l, x_m, y_m, z_m$  of all the points of the quintuple  $< ijklm >$ , that turns out to be equal to nine. Thereby it is proved that for each three-dimensional geometry there exists some equation (2.21) that expresses its phenomenological symmetry. The explicit form of



that equation is found for all the three-dimensional geometries, except for the simplicial and Helmholtz spaces: (2.30), (2.31) and (2.34), (2.35), (2.36). For all the other geometries the equation (2.21) is written in the form of the vanishing to zero of either the Cayly-Menger determinant of the sixth order or of the gramian of the fifth order whose diagonal elements are the values of the metric function for the diagonal pairs  $\langle ii \rangle, \langle jj \rangle, \langle kk \rangle, \langle ll \rangle, \langle mm \rangle$ .

The group symmetry of the three-dimensional geometries, just as in case of the two-dimensional ones, is equivalent to the phenomenological symmetry, according Theorem 3 of §1. The locally invertible transformation

$$x' = \lambda(x, y, z), \quad y' = \sigma(x, y, z), \quad z' = \tau(x, y, z)$$

satisfying the condition  $\partial(\lambda, \sigma, \tau)/\partial(x, y, z) \neq 0$ , will be a local motion if it preserves the metric function (2.20):

$$f(\lambda(i), \sigma(i), \tau(i), \lambda(j), \sigma(j), \tau(j)) = f(x_i, y_i, z_i, x_j, y_j, z_j),$$

where, for example,  $\lambda(i) = \lambda(x_i, y_i, z_i)$ . The solution of that functional equation for each of the three-dimensional geometries (2.23)–(2.37) yields the complete local six-parameter group of motions, and it is that that defines its group symmetry of degree six. The metric function (2.20) is also a solution of the differential equation (2.19) with the operators

$$X = \lambda(x, y, z)\partial/\partial x + \sigma(x, y, z)\partial/\partial y + \tau(x, y, z)\partial/\partial z$$

of the respective six-dimensional Lie algebra. Besides, it is possible to consider that equation as a functional one for the coefficients  $\lambda, \sigma, \tau$  of the operator  $X$ . Interpreted this way, and with the known metric function (2.20), the equation (2.19) is solved by easy enough methods (see [14]). And by the known group of motions of the three-dimensional geometry or its six-dimensional Lie algebra the original metric function is, as a two-point invariant, reconstructed uniquely with an accuracy up to a scaling transformation  $\psi(f) \rightarrow f$ .

We shall note in conclusion that phenomenologically symmetric geometries of higher dimensionality ( $s = 1, n > 3$ ) are defined on an  $n$ -dimensional

manifold by the nondegenerate one-component function with the coordinate representation

$$f(ij) = f(x_i^1, x_i^2, \dots, x_i^n, x_j^1, x_j^2, \dots, x_j^n),$$

the respective set of values  $f(ij), f(ik), \dots, f(vw)$  for the cortege  $\langle ijk \dots vw \rangle$  of length  $n + 2$  and some neighbourhood of it, such that the set of pairs  $\langle ij \rangle, \langle ik \rangle, \dots, \langle vw \rangle$  belongs to its domain, being functionally related by some equation (1.1). Classifications of such geometries have not been built yet, however we can write some expressions for the metric function of the  $n$ -dimensional phenomenologically symmetric geometry of rank  $n + 2$  as natural and special extensions of some certain expressions of a classification of geometries of smaller dimensionality. For example, for the four-dimensional phenomenologically symmetric geometry ( $s = 1, n = 4$ ) of rank 6, the classification, only preliminary and far from being complete, will be, with an accuracy up to a change of the local coordinates  $x, y, z, t$  in the manifold and a scaling transformation  $\psi(f) \rightarrow f$ , (see [15]) as follows:

$$f(ij) = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 + (t_i - t_j)^2, \quad (2.38)$$

$$f(ij) = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 - (t_i - t_j)^2, \quad (2.39)$$

$$f(ij) = (x_i - x_j)^2 + (y_i - y_j)^2 - (z_i - z_j)^2 - (t_i - t_j)^2, \quad (2.40)$$

$$f(ij) = [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2] \exp 2(t_i + t_j), \quad (2.41)$$

$$f(ij) = [(x_i - x_j)^2 + (y_i - y_j)^2 - (z_i - z_j)^2] \exp 2(t_i + t_j), \quad (2.42)$$

$$\begin{aligned} f(ij) = \sin t_i \sin t_j [ & \sin z_i \sin z_j (\sin y_i \sin y_j \cos(x_i - x_j) + \\ & + \cos y_i \cos y_j) + \cos z_i \cos z_j ] + \cos t_i \cos t_j, \end{aligned} \quad (2.43)$$

$$f(ij) = \text{cht}_i \text{cht}_j [\sin z_i \sin z_j (\sin y_i \sin y_j \cos(x_i - x_j) + \cos y_i \cos y_j) + \cos z_i \cos z_j] - \text{sht}_i \text{sht}_j, \quad (2.44)$$

$$f(ij) = \text{sht}_i \text{sht}_j [\sin z_i \sin z_j (\sin y_i \sin y_j \cos(x_i - x_j) + \cos y_i \cos y_j) + \cos z_i \cos z_j] - \text{cht}_i \text{cht}_j, \quad (2.45)$$

$$f(ij) = \text{cht}_i \text{cht}_j [\text{ch} z_i \text{ch} z_j (\sin y_i \sin y_j \cos(x_i - x_j) + \cos y_i \cos y_j) - \text{sh} z_i \text{sh} z_j] - \text{sht}_i \text{sht}_j, \quad (2.46)$$

$$f(ij) = \text{sht}_i \text{sht}_j [\text{ch} z_i \text{ch} z_j (\sin y_i \sin y_j \cos(x_i - x_j) + \cos y_i \cos y_j) - \text{sh} z_i \text{sh} z_j] - \text{cht}_i \text{cht}_j, \quad (2.47)$$

$$f(ij) = x_i y_j - x_j y_i + z_i t_j - z_j t_i, \quad (2.48)$$

$$f(ij) = \frac{(x_i - x_j)^2 \pm (y_i - y_j)^2 \pm (z_i - z_j)^2 + \varepsilon_i t_i^2 + \varepsilon_j t_j^2}{t_i t_j}, \quad (2.49)$$

where  $\varepsilon_i = 0, \pm 1$ ;  $\varepsilon_j = 0, \pm 1$ ,  $\varepsilon_i$  being not necessarily equal to  $\varepsilon_j$ .

We shall note that neither the 4-dimensional simplicial nor Helmholtz spaces are on the list. Apparently, they are not among the phenomenologically symmetric 4-dimensional geometries of rank 6 at all. As to the 4-dimensional geometries that we did manage to come by, the equations expressing their phenomenological symmetry are found quite easily. The respective Cayley-Menger determinants of the seventh order and the sextic Gramians vanish to zero. The group symmetry of degree 10 is determined according to Theorem 3 of §1 by the 10-parameter group of motions that preserve the metric function.

### §3. Dimetric geometries on a plane and trimetric geometries in space

Under the general definitions of §1 the 2-dimensional dimetric geometry ( $s = 2, n = 1$ ) is defined on a two-dimensional manifold  $\mathfrak{M}$  by a two-component metric function  $f = (f^1, f^2)$  that assigns to each pair  $\langle ij \rangle$  from its domain  $\mathfrak{S}_f \subseteq \mathfrak{M} \times \mathfrak{M}$  two real numbers  $f(ij) = (f^1(ij), f^2(ij)) \in R^2$ . The domain  $\mathfrak{S}_f$  of the metric function  $f$  is supposed to be a set open and dense in  $\mathfrak{M} \times \mathfrak{M}$ .

If  $x, y$  are local coordinates in  $\mathfrak{M}$  its coordinate representation

$$f(ij) = f(x_i, y_i, x_j, y_j), \quad (3.1)$$

- or in the more detailed, with respect to the components, writing:

$$\left. \begin{aligned} f^1(ij) &= f^1(x_i, y_i, x_j, y_j), \\ f^2(ij) &= f^2(x_i, y_i, x_j, y_j), \end{aligned} \right\}$$

- is a smooth nondegenerate function of the coordinates  $x_i, y_i$  and  $x_j, y_j$ , that must be included into it in an essential manner. The condition of the metric function  $f = (f^1, f^2)$  being nondegenerate is expressed, mathematically, by two inequalities:

$$\left. \begin{aligned} \partial(f^1(ij), f^2(ij))/\partial(x_i, y_i) &\neq 0, \\ \partial(f^1(ij), f^2(ij))/\partial(x_j, y_j) &\neq 0 \end{aligned} \right\} \quad (3.2)$$

for the set of pairs  $\langle ij \rangle$  open and dense in  $\mathfrak{M} \times \mathfrak{M}$ .

If the two-component metric function (3.1) that satisfies the above three conditions gives on a two-dimensional manifold  $\mathfrak{M}$  a phenomenologically symmetric geometry of rank 3, then for any triple  $\langle ijk \rangle$  of a set dense and open in  $\mathfrak{M}^3$ , such that the pairs  $\langle ij \rangle, \langle ik \rangle, \langle jk \rangle$  belong to  $\mathfrak{S}_f$ , the six reciprocal distances  $f(ij), f(ik), f(jk)$  are functionally related by two independent equations:

$$\Phi(f(ij), f(ik), f(jk)) = 0, \quad (3.3)$$

where  $\Phi = (\Phi_1, \Phi_2)$  is a two-component function of six variables. In a more detailed, in the components, writing, we have:

$$\left. \begin{aligned} \Phi_1(f^1(ij), f^2(ij), f^1(ik), f^2(ik), f^1(jk), f^2(jk)) &= 0, \\ \Phi_2(f^1(ij), f^2(ij), f^1(ik), f^2(ik), f^1(jk), f^2(jk)) &= 0, \end{aligned} \right\}$$

the independence of these equations meaning that  $\text{rang } \Phi = 2$ .

We know very well already that the phenomenological symmetry of a geometry is closely connected with the group symmetry of it. In particular, the plane of thermodynamical states, that we discussed in the Introduction, is, on the one hand, phenomenologically symmetric with rank 3, and on the other - endowed with a group symmetry of degree 2. Such a relation of the rank of one symmetry and of the degree of the other is not occasional and is a corollary of their being equivalent, which is established by Theorem 3 of §1. Indeed, in the case in question, a three-point rigid figure must move freely with two degrees of freedom and not more, because its position is defined by the six coordinates  $x_i, y_i, x_j, y_j, x_k, y_k$ , with the four relations that spring up from the six relations being preserved satisfying the relations (3.3). For the sake of clarity, we shall express the equivalence of the phenomenological and group symmetries of the two-dimensional dimetric geometries with a special theorem:

**Theorem 1.** *For the nondegenerate two-component metric function  $f = (f^1, f^2)$  to give on a two-dimensional manifold  $\mathfrak{M}$  a dimetric phenomenologically symmetric two-dimensional geometry of rank 3 it is necessary and sufficient that it should give on the same manifold a dimetric geometry endowed with group symmetry of degree 2.*

Thus, the two-component metric function (3.1) of a phenomenologically symmetric two-dimensional geometry of rank 3 allows a 2-parameter group of motions, i.e. of such effective smooth local actions in a two-dimensional manifold of some local Lie group  $G^2$ :

$$x' = \lambda(x, y; a^1, a^2), \quad y' = \sigma(x, y; a^1, a^2), \quad (3.4)$$

that each component of the metric function is preserved:

$$f(\lambda(i), \sigma(i), \lambda(j), \sigma(j)) = f(x_i, y_i, x_j, y_j), \quad (3.5)$$

where  $(a^1, a^2) \in G^2$  and, for example,  $\lambda(i) = \lambda(x_i, y_i; a^1, a^2)$ .

We shall write the basic vector fields  $X_1, X_2$  of the two-dimensional Lie algebra of the local transformations (3.4) of the two-dimensional manifold

$\mathfrak{M}$  in the operator from:

$$\left. \begin{aligned} X_1 &= \lambda_1(x, y)\partial_x + \sigma_1(x, y)\partial_y, \\ X_2 &= \lambda_2(x, y)\partial_x + \sigma_2(x, y)\partial_y, \end{aligned} \right\} \quad (3.6)$$

where  $\partial_x = \partial/\partial x$ ,  $\partial_y = \partial/\partial y$  and, for example,  $\lambda_1(x, y) = \partial\lambda(x, y; a^1, a^2)/\partial a^1|_{a^1=a^2=0}$ , supposing that with  $a^1 = a^2 = 0$  we have an identity substitution in the group (3.4). The metric function (3.1), which is, in force of the equality (3.5), the two-point invariant of the group of transformations (3.4), is necessarily a solution of the system of two differential equations as follows:

$$\left. \begin{aligned} X_1(i)f(ij) + X_1(j)f(ij) &= 0, \\ X_2(i)f(ij) + X_2(j)f(ij) &= 0. \end{aligned} \right\} \quad (3.7)$$

with the operators (3.6).

In 1893, Sophus Lie gave a complete classification of the finite dimensional local groups of transformations of the two-dimensional manifold [16]. Out of his classification, it is possible to single out and write, in a suitably chosen system of local coordinates  $(x, y)$ , the basic operators (3.6) of the four respective two-dimensional Lie algebras:

$$X_1 = \partial_x, \quad X_2 = y\partial_x; \quad (3.8)$$

$$X_1 = \partial_x, \quad X_2 = \partial_y; \quad (3.9)$$

$$X_1 = \partial_x, \quad X_2 = x\partial_x; \quad (3.10)$$

$$X_1 = \partial_x, \quad X_2 = x\partial_x + \partial_y. \quad (3.11)$$

**Theorem 2.** *There are two and only two irreducible to each other two-component metric functions  $f = (f^1, f^2)$  that give on a two-dimensional manifold  $\mathfrak{M}$  a phenomenologically symmetric geometry of rank 3. With an accuracy up to a scaling transformation  $\psi(f) \rightarrow f$ , where  $\psi = (\psi_1, \psi_2)$ , and in a suitably chosen system of local coordinates  $(x, y)$  these metric functions may be represented by the two canonical expressions as follows:*

$$f^1(ij) = x_i - x_j, \quad f^2(ij) = y_i - y_j; \quad (3.12)$$

$$f^1(ij) = (x_i - x_j)y_i, \quad f^2(ij) = (x_i - x_j)y_j. \quad (3.13)$$

The components of the metric function  $f = (f^1, f^2)$  are independent solutions of the system of equations (3.7). Since in these equations with the operators (3.8) and (3.10) there is missing an operator of differentiation  $\partial/\partial y_i$ , their independent solutions will be the functions  $y_i$  and  $\varphi(ij)$ . I.e. we get for the metric function the expression  $f(ij) = \psi(y_i, \varphi(ij))$ , where  $\psi : R^2 \rightarrow R^2$ . But it does not satisfy the second of the conditions (3.2), so it turns out to be a degenerate one. The solutions of the system (3.7) with the operators (3.9) are found easily by the method of characteristics, and they coincide in their explicit coordinate representation with the components of the metric function (3.12). The solutions of the system (3.7) with the operators (3.11) are found as easily, but they only coincide in their explicit coordinate representation with the components of the metric function (3.13) on condition that the additional change of coordinates  $x \rightarrow x, \exp(-y) \rightarrow y$  is introduced. Both metric functions are nondegenerate ones, because each of the two Jacobians of theirs in the condition (3.2) is unequal to zero.

The metric function (3.12) may be interpreted, for example, by the projections of the vector  $\vec{j}i$  on the coordinate axes. The corresponding functional relation (3.3) for it is defined by two independent equations

$$\left. \begin{aligned} f^1(ij) - f^1(ik) + f^1(jk) &= 0, \\ f^2(ij) - f^2(ik) + f^2(jk) &= 0. \end{aligned} \right\}$$

The metric function (3.13) allows the essential physical interpretation in thermodynamics discussed in detail in the Introduction. The corresponding functional relation (3.3) for it is defined by the two independent equations

$$\left. \begin{aligned} \begin{vmatrix} 0 & -f^2(ij) & -f^2(ik) \\ f^1(ij) & 0 & -f^2(jk) \\ f^1(ik) & f^1(jk) & 0 \end{vmatrix} &= 0, \\ \begin{vmatrix} f^1(ij) & f^1(jk) & -f^2(ik) \\ f^1(ik) & 0 & -f^2(jk) \\ f^1(ik) & -f^2(ij) & -f^2(jk) \end{vmatrix} &= 0. \end{aligned} \right\}$$

Further we shall discuss trimetric phenomenologically symmetric geometries of rank 3 defined on a three-dimensional manifold  $\mathfrak{M}$  by the three- component

metric function  $f = (f^1, f^2, f^3)$  which assigns to each pair  $\langle ij \rangle$  of its domain  $\mathfrak{S}_f \subseteq \mathfrak{M} \times \mathfrak{M}$  three numbers  $f(ij) = (f^1(ij), f^2(ij), f^3(ij)) \in R^3$ .

Let  $(x, y, z)$  be local coordinates in  $\mathfrak{M}$ . For the metric function  $f$ , in some neighbourhood of the pair  $\langle ij \rangle \in \mathfrak{S}_f$ , it is possible to write down its smooth coordinate representation:

$$f(ij) = f(x_i, y_i, z_i, x_j, y_j, z_j). \quad (3.14)$$

The nondegeneracy of the metric function (3.14), in particular its essential dependence on the coordinates  $x_i, y_i, z_i$  and  $x_j, y_j, z_j$  of the points  $i$  and  $j$ , means nonvanishing to zero of two Jacobians of third order:

$$\left. \begin{aligned} \partial(f^1(ij), f^2(ij), f^3(ij))/\partial(x_i, y_i, z_i) &\neq 0, \\ \partial(f^1(ij), f^2(ij), f^3(ij))/\partial(x_j, y_j, z_j) &\neq 0 \end{aligned} \right\} \quad (3.15)$$

for the set of pairs  $\langle ij \rangle$  open and dense in  $\mathfrak{M} \times \mathfrak{M}$ .

If the three-component metric function (3.14) gives on a three-dimensional manifold  $\mathfrak{M}$  a phenomenologically symmetric geometry of rank 3, then there exists a three-component function  $\Phi = (\Phi_1, \Phi_2, \Phi_3)$  of nine variables, such that the nine reciprocal distances among the points of the set of triples  $\langle ijk \rangle$  open and dense in  $\mathfrak{M}^3$  are functionally related by three independent equations

$$\Phi(f(ij), f(ik), f(jk)) = 0. \quad (3.16)$$

The metric function (3.14) allows a three-parameter group of motions:

$$\left. \begin{aligned} x' &= \lambda(x, y, z; a^1, a^2, a^3), \\ y' &= \sigma(x, y, z; a^1, a^2, a^3), \\ z' &= \tau(x, y, z; a^1, a^2, a^3), \end{aligned} \right\} \quad (3.17)$$

with respect to which it is a nondegenerate two-point invariant and satisfies the functional equation as follows:

$$f(\lambda(i), \sigma(i), \tau(i), \lambda(j), \sigma(j), \tau(j))) = f(x_i, y_i, z_i, x_j, y_j, z_j), \quad (3.18)$$

where, for example,  $\lambda(i) = \lambda(x_i, y_i, z_i; a^1, a^2, a^3)$ . As a corollary of the local invertibility of the transformations (3.17), the condition must hold as follows:

$$\partial(\lambda, \sigma, \tau)/\partial(x, y, z) \neq 0.$$



We shall designate by

$$\left. \begin{aligned} X_1 &= \lambda_1(x, y, z)\partial_x + \sigma_1(x, y, z)\partial_y + \tau_1(x, y, z)\partial_z, \\ X_2 &= \lambda_2(x, y, z)\partial_x + \sigma_2(x, y, z)\partial_y + \tau_2(x, y, z)\partial_z, \\ X_3 &= \lambda_3(x, y, z)\partial_x + \sigma_3(x, y, z)\partial_y + \tau_3(x, y, z)\partial_z \end{aligned} \right\} \quad (3.19)$$

the basic operators of the three-dimensional Lie algebra of the group (3.17). Then, for the metric function (3.14), as a two-point invariant, we have, from the functional equation (3.18), a system of three linear homogeneous differential equations in partial derivatives:

$$\left. \begin{aligned} X_1(i)f(ij) + X_1(j)f(ij) &= 0, \\ X_2(i)f(ij) + X_2(j)f(ij) &= 0, \\ X_3(i)f(ij) + X_3(j)f(ij) &= 0 \end{aligned} \right\} \quad (3.20)$$

with the operators (3.19).

Thus, the task of the classification of the metric functions (3.14) narrows down to that of classification of the three-dimensional Lie algebras of the transformations of the three-dimensional manifold with the basic operators (3.19) and to that of integrating the respective systems of equations (3.20). It is quite easy to make sure that the solution of the system will only give a nondegenerate metric function if the group of transformations (3.17) is transitive, for which, as is known, it is necessary and sufficient that the rank of the matrix of the coefficients of the operators (3.19) be equal to 3.

**Theorem 3.** *The basic operators (3.19) of the three-dimensional Lie algebra of the local Lie group of the locally transitive transformations of a three-dimensional manifold with an accuracy up to isomorphism and in a suitably chosen system of local coordinates  $(x, y, z)$  are defined by the expressions as follows:*

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = \partial_z; \quad (3.21)$$

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = y\partial_x + \partial_z; \quad (3.22)$$

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = (x + y)\partial_x + y\partial_y + \partial_z; \quad (3.23)$$

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x\partial_x + py\partial_y + \partial_z; \quad (3.24)$$

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = -y\partial_x + (x + qy)\partial_y + \partial_z; \quad (3.25)$$

$$\left. \begin{aligned} X_1 &= \partial_x, \\ X_2 &= \operatorname{tg} y \sin x \partial_x + \cos x \partial_y + \sec y \sin x \partial_z, \\ X_3 &= \operatorname{tg} y \cos x \partial_x - \sin x \partial_y + \sec y \cos x \partial_z; \end{aligned} \right\} \quad (3.26)$$

$$\left. \begin{aligned} X_1 &= \partial_x, \\ X_2 &= \sin x \partial_x + \cos x \partial_y + \exp y \sin x \partial_z, \\ X_3 &= \cos x \partial_x - \sin x \partial_y + \exp y \cos x \partial_z, \end{aligned} \right\} \quad (3.27)$$

where  $-1 \leq p \leq 1$ ,  $0 \leq q < 2$ .

The classification in this Theorem 3 was built up by the author and can be found in his note [17].

**Theorem 4.** *With an accuracy up to a scaling transformation  $\psi(f) \rightarrow f$ , where  $\psi = (\psi_1, \psi_2, \psi_3)$ , and in a suitably chosen system of local coordinates  $(x, y, z)$  the metric function  $f = (f^1, f^2, f^3)$  that defines on a three-dimensional manifold  $\mathfrak{M}$  a phenomenologically symmetric geometry of rank 3 may be represented by one of the following eleven canonical expressions:*

$$f^1(ij) = x_i - x_j, \quad f^2(ij) = y_i - y_j, \quad f^3(ij) = z_i - z_j; \quad (3.28)$$

$$\left. \begin{aligned} f^1(ij) &= y_i - y_j, \\ f^2(ij) &= (x_i - x_j)y_i + z_i - z_j, \\ f^3(ij) &= (x_i - x_j)y_j + z_i - z_j; \end{aligned} \right\} \quad (3.29)$$

$$\left. \begin{aligned} f^1(ij) &= (x_i - x_j)^2 \exp \left( 2 \frac{y_i - y_j}{x_i - x_j} \right), \\ f^2(ij) &= (x_i - x_j)z_i, \quad f^3(ij) = (x_i - x_j)z_j; \end{aligned} \right\} \quad (3.30)$$

$$\left. \begin{aligned} f^1(ij) &= \frac{y_i - y_j}{x_i - x_j}, \\ f^2(ij) &= (x_i - x_j)z_i, \\ f^3(ij) &= (x_i - x_j)z_j; \end{aligned} \right\} \quad (3.31)$$

$$\left. \begin{aligned} f^1(ij) &= (x_i - x_j)(y_i - y_j), \\ f^2(ij) &= (x_i - x_j)z_i, \\ f^3(ij) &= (x_i - x_j)z_j; \end{aligned} \right\} \quad (3.32)$$

$$\left. \begin{aligned} f^1(ij) &= y_i - y_j, \\ f^2(ij) &= (x_i - x_j)z_i, \\ f^3(ij) &= (x_i - x_j)z_j; \end{aligned} \right\} \quad (3.33)$$

$$\left. \begin{aligned} f^1(ij) &= \frac{(x_i - x_j)^p}{y_i - y_j}, \\ f^2(ij) &= (x_i - x_j)z_i, \\ f^3(ij) &= (x_i - x_j)z_j; \end{aligned} \right\} \quad (3.34)$$

$$\left. \begin{aligned} f^1(ij) &= (x_i - x_j)^2 + (y_i - y_j)^2, \\ f^2(ij) &= z_i + \operatorname{arctg} \left( \frac{y_i - y_j}{x_i - x_j} \right), \\ f^3(ij) &= z_j + \operatorname{arctg} \left( \frac{y_i - y_j}{x_i - x_j} \right); \end{aligned} \right\} \quad (3.35)$$

$$\left. \begin{aligned} f^1(ij) &= ((x_i - x_j)^2 + (y_i - y_j)^2) \times \\ &\quad \times \exp \left( 2\gamma \operatorname{arctg} \frac{y_i - y_j}{x_i - x_j} \right), \\ f^2(ij) &= z_i + \operatorname{arctg} \left( \frac{y_i - y_j}{x_i - x_j} \right), \\ f^3(ij) &= z_j + \operatorname{arctg} \left( \frac{y_i - y_j}{x_i - x_j} \right); \end{aligned} \right\} \quad (3.36)$$

$$\left. \begin{aligned} f^1(ij) &= \sin y_i \sin y_j \cos(x_i - x_j) + \cos y_i \cos y_j, \\ f^2(ij) &= z_i - \arcsin \left( \frac{\sin(x_i - x_j) \sin y_j}{\sqrt{1 - (f^1(ij))^2}} \right), \\ f^3(ij) &= z_j + \arcsin \left( \frac{\sin(x_i - x_j) \sin y_i}{\sqrt{1 - (f^1(ij))^2}} \right); \end{aligned} \right\} \quad (3.37)$$

$$\left. \begin{aligned} f^1(ij) &= (x_i - x_j)y_i y_j, \\ f^2(ij) &= z_i + \frac{1}{(x_i - x_j)y_i^2}, \\ f^3(ij) &= z_j - \frac{1}{(x_i - x_j)y_j^2}, \end{aligned} \right\} \quad (3.38)$$

herein  $0 < |p| < 1$  u  $0 < \gamma < \infty$ , where  $\gamma = q/\sqrt{4 - q^2}$  with  $0 < q < 2$ .

The detailed proof of Theorem 4 is given in §5 of the author's monograph [10]. Essentially, it is the solution, one after another, of all the systems of equations (3.20) with the operators (3.21)–(3.27), which does not pose any technical difficulties. In the final form of the canonical expressions given above, in some places a suitable change of coordinates was performed and the cases of  $p = 0, \pm 1$ ;  $\gamma = 0$  were singled out.

The phenomenological symmetry of all the eleven geometries defined by the metric functions (3.28)–(3.38) is established by the rank of the functional matrix for the nine functions  $f(ij), f(ik), f(jk)$  that depend in special manner on nine variables - the coordinates of the points of the triple  $\langle ij k \rangle$ , and that rank turns out to be equal to 6. The respective functional equations (3.16) that express that symmetry may be found in the explicit form, which is the essence of the following theorem, that was proved by R.M. Muradov in his note (in print).

**Theorem 5.** *If a three-component metric function (3.14) gives on a three-dimensional manifold  $\mathfrak{M}$  a phenomenologically symmetric geometry of rank 3, then with an accuracy up to a change of local coordinates in the manifold and a scaling transformation  $\psi(f) \rightarrow f$ , where  $\psi = (\psi_1, \psi_2, \psi_3)$ , it defines in  $R^3$  a local quazigroup operation with a right identity such that the right inverse coincides with the original one and the equation that expresses the phenomenological symmetry has a form similar to that of the metric function itself:*

$$f(ij) = f(f^1(ik), f^2(ik), f^3(ik), f^1(jk), f^2(jk), f^3(jk)). \quad (3.39)$$

In the note pointed out, for each of the metric functions (3.28)–(3.38) there are coordinate representations corresponding Theorem 5 discovered such that make it possible to write the equation (3.16) by the formula (3.39) in the explicit form.

The components of the metric function (3.28) may be interpreted by way of the projections of the vector  $\vec{j}i$  onto the coordinate axes. The corresponding functional relation (3.16) is defined by the system of three independent equations:

$$\left. \begin{aligned} f^1(ij) - f^1(ik) + f^1(jk) &= 0, \\ f^2(ij) - f^2(ik) + f^2(jk) &= 0, \\ f^3(ij) - f^3(ik) + f^3(jk) &= 0. \end{aligned} \right\}$$

The metric function (3.29) allows an essential physical interpretation in thermodynamics. Let us consider the first component of it the difference between the temperatures  $T_i$  and  $T_j$  of a thermodynamic system in the states  $i$  and  $j$ , and the second and third - works  $A^{TS}(ij)$  and  $A^{ST}(ij)$  done by outward bodies when transferring the system from the state  $i$  to the state  $j$ , along the two-way process, i.e. comprised of an equilibrium isothermic ( $T = \text{const}$ ) and adiabatic ( $S = \text{const}$ ) processes:

$$\left. \begin{aligned} f^1(ij) &= T_i - T_j, \\ f^2(ij) &= A^{TS}(ij) = (S_i - S_j)T_i - U_i + U_j, \\ f^3(ij) &= A^{ST}(ij) = (S_i - S_j)T_j - U_i + U_j, \end{aligned} \right\}$$

where  $S$ ,  $T$  and  $U$  are respectively the entropy, temperature, and internal energy of the system. The corresponding functional relation (3.16) is defined by three independent equations:

$$\left. \begin{aligned} f^1(ij) - f^1(ik) + f^1(jk) &= 0, \\ \frac{f^2(ij) - f^3(ij)}{f^1(ij)} - \frac{f^2(ik) - f^3(ik)}{f^1(ik)} + \frac{f^2(jk) - f^3(jk)}{f^1(jk)} &= 0, \\ \frac{f^3(ij) - f^3(ik) + f^2(jk)}{f^1(jk)} - \frac{f^2(ik) - f^3(ik)}{f^1(ik)} &= 0. \end{aligned} \right\}$$

Moreover, it is also possible to attach essential thermodynamic interpretations to the components of the metric function (3.33), by way of the temperature difference and the works of the mediums done to the system in turning it from the state  $i$  to the state  $j$  along the ways  $PV$  and  $VP$ , where  $P$  is the pressure and  $V$  the volume of the system. The question of interpretation of other trimetric geometries is still open. Their nontrivial symmetries, the

group and the phenomenological ones, that condition each other, give some grounds for hopes that such interpretations will be found for other metric functions of the classification list (3.28)–(3.38).

By now, V.A. Kyrov has built a classification of four-metric (s=4, n=1) phenomenologically symmetric geometries of rank 3 that we shall give after his note [18]:

**Theorem 6.** *With an accuracy up to a scaling transformation  $\psi(f) \rightarrow f$  where  $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$  in a suitably chosen system of local coordinates  $(x, y, z, t)$ , the metric function  $f = (f^1, f^2, f^3, f^4)$  that defines on a four-dimensional manifold  $\mathfrak{M}$  a phenomenologically symmetric geometry of rank 3 may be represented **explicitly** by one of the following twelve canonical expressions:*

$$\left. \begin{aligned} f^1(ij) &= (x_i - x_j)^2 \exp[\varepsilon(t_i + t_j)], \quad f^2(ij) = (y_i - y_j)^2 \exp[k(t_i + t_j)], \\ f^3(ij) &= (z_i - z_j)^2 \exp[l(t_i + t_j)], \quad f^4(ij) = t_i - t_j; \end{aligned} \right\}$$

$$\left. \begin{aligned} f^1(ij) &= [(x_i - x_j)^2 + (y_i - y_j)^2] \exp \left( -2k \operatorname{arctg} \frac{y_i - y_j}{x_i - x_j} \right), \\ f^2(ij) &= 2 \operatorname{arctg} \frac{y_i - y_j}{x_i - x_j} + t_i + t_j, \\ f^3(ij) &= (z_i - z_j)^2 \exp[l(t_i + t_j)], \quad f^4(ij) = t_i - t_j; \end{aligned} \right\}$$

$$\left. \begin{aligned} f^1(ij) &= (x_i - x_j)^2 \exp \left( -2k \frac{y_i - y_j}{x_i - x_j} \right), \quad f^2(ij) = 2 \frac{y_i - y_j}{x_i - x_j} + t_i + t_j, \\ f^3(ij) &= (z_i - z_j)^2 \exp[\varepsilon(t_i + t_j)], \quad f^4(ij) = t_i - t_j; \end{aligned} \right\}$$

$$\left. \begin{aligned} f^1(ij) &= x_i - x_j, \quad f^2(ij) = 2 \frac{y_i - y_j}{x_i - x_j} - (t_i + t_j), \\ f^3(ij) &= z_i - z_j - \frac{(y_i - y_j)^2}{2(x_i - x_j)}, \quad f^4(ij) = t_i - t_j; \end{aligned} \right\}$$

$$\left. \begin{aligned} f^1(ij) &= x_i - x_j, \quad f^2(ij) = 2 \frac{y_i - y_j}{x_i - x_j} - (t_i + t_j), \\ f^3(ij) &= (x_i - x_j) \ln(z_i - z_j + y_i - y_j + x_i - x_j) - y_i + y_j, \\ f^4(ij) &= t_i - t_j; \end{aligned} \right\}$$

$$\left. \begin{aligned} f^1(ij) &= (x_i - x_j)^2 \exp\left(-2k \frac{y_i - y_j}{x_i - x_j}\right), \quad f^2(ij) = 2 \frac{y_i - y_j}{x_i - x_j} - (t_i + t_j), \\ f^3(ij) &= k(y_i - y_j) - (x_i - x_j) - k^2(z_i - z_j), \quad f^4(ij) = t_i - t_j; \end{aligned} \right\}$$

$$\left. \begin{aligned} f^1(ij) &= (x_i - x_j)^2 \exp\left(-2k \frac{y_i - y_j}{x_i - x_j}\right), \quad f^2(ij) = 2 \frac{y_i - y_j}{x_i - x_j} - (t_i + t_j), \\ f^3(ij) &= 2 \frac{z_i - z_j}{x_i - x_j} - k \left( \frac{y_i - y_j}{x_i - x_j} \right)^2, \quad f^4(ij) = t_i - t_j; \end{aligned} \right\}$$

$$\left. \begin{aligned} f^1(ij) &= (x_i - x_j - z_i(y_i - y_j))^2 \exp[c(t_i + t_j)], \\ f^2(ij) &= (x_i - x_j - z_j(y_i - y_j))^2 \exp[c(t_i + t_j)], \\ f^3(ij) &= (y_i - y_j)^2 \exp(t_i + t_j), \quad f^4(ij) = t_i - t_j; \end{aligned} \right\}$$

$$\left. \begin{aligned} f^1(ij) &= (x_i - x_j)e^{z_i}, \quad f^2(ij) = (x_i - x_j)e^{z_j}, \\ f^3(ij) &= (y_i - y_j)e^{t_i}, \quad f^4(ij) = (y_i - y_j)e^{t_j}; \end{aligned} \right\}$$

$$\left. \begin{aligned} f^1(ij) &= [(x_i - x_j)^2 + (y_i - y_j)^2] \exp(z_i + z_j), \\ f^2(ij) &= 2 \arctg \frac{y_i - y_j}{x_i - x_j} + t_i + t_j, \quad f^3(ij) = z_i - z_j, \quad f^4(ij) = t_i - t_j; \end{aligned} \right\}$$

$$\left. \begin{aligned} f^1(ij) &= \sin y_i \sin y_j \cos(x_i - x_j) + \cos y_i \cos y_j, \\ f^2(ij) &= z_i - \arcsin \frac{\sin(x_i - x_j) \sin y_j}{\sqrt{1 - (f^1(ij))^2}}, \\ f^3(ij) &= z_j + \arcsin \frac{\sin(x_i - x_j) \sin y_i}{\sqrt{1 - (f^1(ij))^2}}, \\ f^4(ij) &= t_i - t_j; \end{aligned} \right\}$$

$$\left. \begin{aligned} f^1(ij) &= (x_i - x_j)y_i y_j, \quad f^2(ij) = z_i + \frac{1}{(x_i - x_j)y_i^2}, \\ f^3(ij) &= z_j - \frac{1}{(x_i - x_j)y_j^2}, \quad f^4(ij) = t_i - t_j; \end{aligned} \right\}$$

and, **implicitly**, by two more expressions:

$$\left. \begin{aligned} f^1(ij) &= f^1(x_i - x_j - z_i(y_i - y_j), x_i - x_j - z_j(y_i - y_j), y_i - y_j, t_i, t_j), \\ f^2(ij) &= f^2(x_i - x_j - z_i(y_i - y_j), x_i - x_j - z_j(y_i - y_j), y_i - y_j, t_i, t_j), \\ f^3(ij) &= f^3(x_i - x_j - z_i(y_i - y_j), x_i - x_j - z_j(y_i - y_j), y_i - y_j, t_i, t_j), \\ f^4(ij) &= t_i - t_j, \end{aligned} \right\}$$

wherein the four components of the function  $f = f(u, v, w, t_i, t_j)$  are independent integrals of either the equation

$$\left( qu - \frac{1}{2}v^2 + \frac{1}{2} \left( \frac{v-u}{w} \right)^2 \right) \frac{\partial f}{\partial u} + \left( qu + \frac{1}{2}v^2 - \frac{1}{2} \left( \frac{v-u}{w} \right)^2 \right) \frac{\partial f}{\partial v} - \frac{v-u}{w} \frac{\partial f}{\partial w} + \frac{\partial f}{\partial t_i} + \frac{\partial f}{\partial t_j} = 0,$$

or of the equation

$$\left( 2u - \frac{1}{2} \left( \frac{v-u}{w} \right)^2 \right) \frac{\partial f}{\partial u} + \left( 2v + \frac{1}{2} \left( \frac{v-u}{w} \right)^2 \right) \frac{\partial f}{\partial v} + \left( 2v + \frac{v-u}{w} \right) \frac{\partial f}{\partial w} + \frac{\partial f}{\partial t_i} + \frac{\partial f}{\partial t_j} = 0,$$

where  $k, l, c, q$  are arbitrary numbers, and  $\varepsilon = 0, 1$ .

The same, generally, may be said about that classification as about that given by Theorem 4. In particular, the phenomenological symmetry of four-dimensional geometries is established by the rank of the functional matrix for the twelve components  $f(ij), f(ik), f(jk)$  of the metric function  $f$  that depend in a special manner on 12 variables, the coordinates of the triple  $\langle ijk \rangle$ , which rank equals 8. Some four-component metric functions have an essential physical interpretation.

Classifying of  $s$ -metric phenomenologically symmetric geometries of rank three for  $s > 4$  has not been attempted by anyone, because of the difficulties of sheerly technical nature. They are, for the most part, consequence of the method employed, the essence of which is in classifying first the  $s$ -dimensional Lie groups of the transformations of the space  $R^s$ , only then followed by the finding of the metric functions as nondegenerate two-point invariants. It is possible these difficulties will be overcome if, in classifying groups of transformations, an outright condition be introduced of the existence of nondegenerate two-point invariants for them.



#### §4. The symmetry of distance in geometry

Ordinarily, the distance between points of a space  $\mathfrak{M}$  is determined by the function  $\rho : \mathfrak{M} \times \mathfrak{M} \rightarrow R$  that assigns to each pair of points  $\langle ij \rangle$  some number  $\rho(ij)$  and satisfies the axioms as follows: **1.**  $\rho(ij) \geq 0$ ,  $\rho(ij)$  being equal to 0 if and only if  $i = j$ ; **2.**  $\rho(ij) = \rho(ji)$ ; **3.**  $\rho(ik) + \rho(jk) \geq \rho(ij)$ . Under the axiom **2.** distance is symmetric. However, the so called symplectic spaces, where the distance between two points is antisymmetric are not to be wiped off the slate of geometry either. On the other hand, the symmetric interval between events in the pseudo-Euclidean space-time of Minkowski, that does not satisfy the axioms **1.** and **3.**, can also be considered as a distance. The question that naturally and logically suggests itself is why only symmetric and antisymmetric distances are allowed. It appears that if the existence of a functional relation between the distances  $\rho(ij)$  and  $\rho(ji)$  is assumed, then only these two types of symmetry will be possible. Let us now move on to exact formulations.

Let  $\mathfrak{M} = \{i, j, k, \dots\}$  be a set of arbitrary nature and  $f : G_f \rightarrow R$  a function where  $G_f \subseteq \mathfrak{M} \times \mathfrak{M}$  that assigns to an ordered pair  $\langle ij \rangle \in G_f$  a real number  $f(ij) \in R$  considered as a distance, in some generalized sense. We shall call the two-point function  $f$  a metric one and not require it should satisfy ordinary metric axioms. In a general case the domain  $G_f$  of the function  $f$  does not necessarily coincide with the whole of the direct product  $\mathfrak{M} \times \mathfrak{M}$ . But it is natural to suppose that if  $\langle ij \rangle \in G_f$ , then  $\langle ji \rangle \in G_f$  too, i.e. the distances  $f(ij)$  and  $f(ji)$  are simultaneously either defined or undefined.

**Definition.** We shall say that the metric functions  $f$  and  $g$  are equivalent if their domains  $G_f$  and  $G_g$  coincide in  $\mathfrak{M} \times \mathfrak{M}$  and there exists a strictly monotone function  $\psi : f(G_f) \rightarrow R$  such that for any pair  $\langle ij \rangle \in G_f$  the equality  $g(ij) = \psi(f(ij))$  takes place.

Basing on the remark in the note [2], we shall formulate the symmetry axiom as follows [19]:

**A.S.** For any points  $i, j \in \mathfrak{M}$  such that the pairs  $\langle ij \rangle$  and  $\langle ji \rangle$

belong to  $G_f$ , the distances  $f(ij)$  and  $f(ji)$  are tied by the relation

$$f(ij) = \Theta(f(ji)), \quad (4.1)$$

where  $\Theta$  is some strictly monotone function of one variable whose domain and range of values coincide with the domain  $f(G_f)$  of the original metric function.

**Theorem.** *If the distance between points of a space  $\mathfrak{M}$  determined by the metric function  $f : G_f \rightarrow R$ , where  $G_f \subseteq \mathfrak{M} \times \mathfrak{M}$ , satisfies the symmetry axiom **A.S.**, then that distance may only be either symmetric or, with an accuracy up to equivalence, antisymmetric.*

Out of the relation (4.1) for any pair  $\langle ij \rangle \in G_f$  we get the identity  $\Theta(\Theta(f(ij))) = f(ij)$  that means that the function  $\Theta$  is a solution of the functional equation

$$\Theta(\Theta(x)) = x, \quad (4.2)$$

where  $x \in f(G_f) \subseteq R$ . By assumption, the function  $\Theta$  is a strictly monotone one and so must have its inverse. If the function  $\Theta$  monotone increasing, then  $\Theta(x) = x$  and the distance turns out to be symmetric. And if the function  $\Theta$  monotone decreasing, then, switching over to the equivalent metric function  $g = \psi(f)$ , where  $\psi(f) = f - \Theta(f)$ , we have, under (4.1),  $g(ij) = f(ij) - \Theta(f(ij)) = f(ij) - f(ji) = -g(ji)$ , i.e. an antisymmetric distance. The theorem has been proved.

Symmetry or antisymmetry of a distance in geometry, with the relation (4.1) present were established by the author in his note "Some consequences of the hypothesis of binary structure of space" [20] for the case where that distance is determined by the function  $F : G_F \rightarrow R$ , where  $G_F \subseteq \mathfrak{M} \times \mathfrak{N}$ , defining on  $n$ -dimensional manifolds  $\mathfrak{M} = \{i, j, k, \dots\}$  and  $\mathfrak{N} = \{\alpha, \beta, \gamma, \dots\}$  a phenomenologically symmetric geometry of two sets (physical structure) of rank  $(n+1, n+1)$  and of some local diffeomorphism  $\varphi : \mathfrak{M} \rightarrow \mathfrak{N}$ :

$$f(ij) = F(i, \varphi(j)). \quad (4.3)$$

For the distance (4.3), the relation (4.1), with the known function  $F$  becomes the functional equation with respect to the function  $\Theta$  and the

diffeomorphism  $\varphi$ :

$$F(i, \varphi(j)) = \Theta(F(j, \varphi(i))).$$

By solving that equation for the functions

$$\begin{aligned} F(i\alpha) &= x_i^1 \xi_\alpha^1 + \cdots + x_i^n \xi_\alpha^n, \\ F(i\alpha) &= x_i^1 \xi_\alpha^1 + \cdots + x_i^{n-1} \xi_\alpha^{n-1} + x_i^n + \xi_\alpha^n, \end{aligned}$$

where  $x^1, \dots, x^n$  and  $\xi^1, \dots, \xi^n$  are local coordinates in the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ , it is possible to find both the diffeomorphism  $\varphi$  and the function  $\Theta$  determining the type of the distance symmetry (4.3). In a suitably chosen in the manifold  $\mathfrak{M}$  system of local coordinates, the expressions for the distance  $f(ij)$  may be written with an accuracy up to a local equivalence as follows:

$$\left. \begin{aligned} f(ij) &= g_{\lambda\sigma} x_i^\lambda x_j^\sigma, \\ f(ij) &= h_{\mu\nu} x_i^\mu x_j^\nu + x_i^n + ax_j^n, \end{aligned} \right\} \quad (4.4)$$

where  $a = +1, -1$ ;  $g_{\lambda\sigma} = ag_{\sigma\lambda}$ ,  $\lambda, \sigma = 1, \dots, n$ ;  $h_{\mu\nu} = ah_{\nu\mu}$ ,  $\mu, \nu = 1, \dots, n-1$ , the dimensionality  $n$  of the manifold  $\mathfrak{M}$  being even for  $a = -1$  in the former expression (4.4) and odd in the latter.

Out of the expressions (4.4), by introducing some natural additional conditions, in the case of  $a = +1$ , we can get symmetric metric functions of Riemannian and pseudo-Riemannian spaces of constant curvature. And in case  $a = -1$  the expressions (4.4) define antisymmetric metric functions of symplectic spaces of *even* and, also, we shall point it out, *odd* dimensionality.

## §5. Binary and ternary geometries

Binary phenomenologically symmetric geometries are defined on one set. A two-point function that gives such a geometry allows a nontrivial group of motions with a finite number of continuous parameters that has been called the degree of a group symmetry. With certain relations among the rank of the phenomenological symmetry, the number of the essential parameters of the group of motions, and the dimensionality of the manifold, the group and phenomenological symmetry turn out to be equivalent. Those relations were

incorporated into the definition of the geometry, and its phenomenological and group symmetries. Naturally, the question arises of their origin and interpretation. Besides, there are a lot of opportunities of generalization and development of the notion of geometry, one of which is realized in §1 when two points are assigned more than one real numbers. Another opportunity for generalization is realized in defining of, for example, ternary geometries, where a metric function assigns a number not to two but to three points. However, as early as on the stage of preliminary exploration it turned out that the ternary geometries, in contrast to binary, may not have a group symmetry, i.e. a three-point metric function does not allow a nontrivial group of motions. So additionally there arises the question of the causes of such difference between the binary and ternary geometries.

To answer these questions it is necessary to proceed from a more general definition of polyary geometries. Then it will be possible to find out at what relations among the basic characteristics of a geometry it may be endowed with a group symmetry, and at what it may not. It is natural to suppose that only those geometries are meaningful, physically and mathematically, whose groups of motions are nontrivial.

Let there be a set  $\mathfrak{M}$  of arbitrary nature that is, mathematically, a smooth manifold of dimension  $m$ . Let also there be a function

$$f : \mathfrak{S}_f \rightarrow R^s, \quad (5.1)$$

where  $\mathfrak{S}_f \subseteq \mathfrak{M}^q$ , that assigns to each cortege of length  $q$  from  $\mathfrak{S}_f$  some point from  $R^s$ , i.e.  $s$  real numbers. It is assumed that the domain  $\mathfrak{S}_f$  of the function  $f$  is open and dense in the  $q$ -ary direct product  $\mathfrak{M}^q$  of the set  $\mathfrak{M}$  by itself, and that its coordinate representation is sufficiently smooth. We shall call the number  $q$  arity, and a  $q$ -ary and  $s$ -component function (5.1) - a metric one.

Let, further,  $M > q$  be an arbitrary integer number. Let us construct the mapping

$$F : \mathfrak{S}_F \rightarrow R^{sC_M^q}, \quad (5.2)$$

where  $\mathfrak{S}_F \subseteq \mathfrak{M}^M$ , by assigning to each cortege of length  $M$  from  $\mathfrak{S}_F$  a collection of numbers  $sC_M^q$  ordered with respect to that cortege, the numbers of which collection correspond to all the corteges of length  $q$ , that

are projections of the original cortege onto the domain  $\mathfrak{S}_f$ . The domain  $\mathfrak{S}_F$  of the function (5.2) is, obviously, open and dense in the direct product  $\mathfrak{M}^M$ . Similarly, we shall build another mapping

$$F' : \mathfrak{S}_{F'} \rightarrow R^{sC_{M'}^q}, \quad (5.2')$$

where  $\mathfrak{S}_{F'} \subseteq \mathfrak{M}^{M'}$  and  $M' \geq M$ . The projection of the mapping  $F'$  is obtained from the domain  $\mathfrak{S}_{F'}$  of it by way of dropping some collection of corteges of length  $q$ , along with the corresponding numbers from the region of its values with respect to the function (5.1).

**Definition 1.** We shall say that the function (5.1) gives on an  $m$ -dimensional manifold  $\mathfrak{M}$  a *q-ary s-metric phenomenologically symmetric geometry of rank  $M$*  if on a set dense in  $\mathfrak{S}_F$  the rank of the mapping  $F$  is equal to  $s(C_M^q - 1)$ , and the rank of any projection of the mapping  $F'$  that does not include the range of the mapping  $F$  is maximal in the set dense in  $\mathfrak{S}_{F'}$ .

In other words, locally the range of the mapping  $F$  in  $R^{sC_M^q}$  belongs to the set of zeros of the system of  $s$  independent functions  $\Phi = (\Phi_1, \dots, \Phi_s)$  of  $sC_M^q$  variables,  $s$  functional relations

$$\Phi = (\Phi_1, \dots, \Phi_s) = 0 \quad (5.3)$$

being generating ones in that particular sense that any other nontrivial relations will be just their consequence.

**Definition 2.** We shall say that the phenomenologically symmetric geometry that we have defined above is *endowed with the group symmetry of finite degree  $r$* , if an effective smooth local action of some  $r$ -dimensional local Lie group  $G^r$  in the manifold  $\mathfrak{M}$  is defined, such that the components of the metric function (5.1) that gives that geometry are  $q$ -point invariants.

Since the manifold transformed is finite-dimensional, Definition 2 naturally implies the condition that the maximum number of the essential parameters of the group of local motions is finite.

We shall write down a system of  $sC_{M'}^q$  equations that are result of the condition of preservation of the components of the metric function (5.1):

$$Df|_{F'} = 0 \quad (5.4)$$

with respect to  $M'm$  differentials of the points of the cortege of  $\mathfrak{S}_{F'}$ . If the phenomenologically symmetric geometry introduced by Definition 1 is endowed with a group symmetry of finite degree, then, on the one hand, the homogeneous system (5.4) must have at least one nonzero solution, and, on the other hand, the number of its independent nonzero solutions for any number  $M'$  may not be bigger than some finite value equal to the degree of the group symmetry. The number of such solutions is, as is known, equal to the number of unknowns in the system minus the rank of matrix of it. But the matrix of the system of equations (5.4) is the functional matrix for the system of functions  $f$  corresponding to all the ordered projections of the domain  $\mathfrak{S}_{F'}$  of the mapping (5.2') onto the domain  $\mathfrak{S}_f$  of the original function (5.1). Obviously, the rank of the matrix of the system of equations (5.4) will not change if we eliminate from the system of functions  $f|_{F'}$  the dependents with respect to the relation (5.3). Eliminating them yields the maximal projection of the mapping (5.2') that does not contain in itself the mapping (5.2). We shall designate by  $N(M')$  the number of functions  $f$  in that maximal projection. Then, under Definition 1, the rank of the matrix of the system of equations (5.4) will be

$$\min(M'm; N(M')). \quad (5.5)$$

If there exists such a value of the number  $M'$ , for which  $M'm \leq N(M')$ , the rank of the matrix of the system of equations (5.4) for it will be equal to  $M'm$ , i.e. to the number of unknowns in it. But then the system (5.4) will only have a zero solution, which means the absence of a nontrivial group of motions in the phenomenologically symmetric geometry in question. But if the strict inequality  $N(M') < M'm$  holds for any value of  $M'$ , then the rank of the matrix of the system of equations (5.4) will be equal to  $N(M')$ , and the number of the linearly independent nonzero solutions of it will be equal to

$$r' = M'm - N(M') > 0. \quad (5.6)$$

The number  $r'$ , as has already been stated, in case of a phenomenologically symmetric geometry having a group symmetry of finite degree may not exceed some finite value. Out of that condition, let us establish at what relation between the dimensionality of the set and the rank of the

phenomenological symmetry the geometry defined under Definition 1 by the metric function (5.1) may have a group symmetry and shall define the degree  $r$  of that symmetry.

Let us take first binary ( $q=2$ ) geometries.

A binary  $s$ -metric phenomenologically symmetric geometry of rank  $M \geq 3$  on one set  $\mathfrak{M}$  that is an  $m$ -dimensional manifold is defined by the metric function (5.1), where  $\mathfrak{S}_f \subseteq \mathfrak{M} \times \mathfrak{M}$ , and, under Definition 1, the rank of the mapping  $F : \mathfrak{S}_F \rightarrow R^{sM(M-1)/2}$ , where  $\mathfrak{S}_F \subseteq \mathfrak{M}^M$ , is equal to  $sM(M-1)/2 - s$ . We shall find the number of the dependents in the system of  $sM'(M'-1)/2$  functions of the mapping  $F' : \mathfrak{S}_{F'} \rightarrow R^{sM'(M'-1)/2}$ , where  $\mathfrak{S}_{F'} \subseteq \mathfrak{M}^{M'}$  and  $M' \geq M$ . We shall sequentially superpose the matrix of the pairs of the cortege of length  $F$  from  $\mathfrak{S}_F$  onto the matrix of the pairs for the cortege of length  $M'$  from  $\mathfrak{S}_{F'}$ . At each complete superposition, we shall cross out one pair, for example, the last one. The procedure is repeated so long as superposition without blanks is possible. The number of the dependent functions will obviously be equal to the number of successful superpositions multiplied by  $s$ . It is readily established that the number is equal to  $s(M'-M+1)(M'-M+2)/2$ , and hence the rank of the functional matrix of the system of functions  $f|_{F'}$ , in accordance with Definition 1, will be equal to

$$\min(M'm; sM'(M'-2)/2 - s(M'-M+1)(M'-M+2)/2).$$

If  $m < s(M-2)$ , then for sufficiently large values of  $M'$  the rank of the matrix of the system of equations (5.4), in the case in question, is equal to  $M'm$ , i.e. to the number of the unknowns in it, and so it only has for those values of  $M'$  a zero solution. Therefore, with  $m < s(M-2)$  a binary  $s$ -metric phenomenologically symmetric geometry of rank  $M$  may not have a group symmetry. But if  $m \geq s(M-2)$ , then, for every  $M' \geq M$ , the rank of the matrix of the system (5.4) is less than  $M'm$ , and it has, according to the expression (5.6),

$$r' = M'm - sM'(M-2) + s(M-1)(M-2)/2$$

linearly independent nonnull solutions. At  $m > s(M-2)$ , with  $M'$  increasing,

the number of solutions  $r'$  may become arbitrarily large, which is in contradiction with the condition, under Definition 2, of the degree of the group symmetry being finite. So, if the binary geometry in question is endowed with a group symmetry of finite degree the dimension  $m$  of the manifold  $\mathfrak{M}$  and its rank must be tied by the relation

$$\mathbf{m} = \mathbf{s}(\mathbf{M} - 2). \quad (5.7)$$

Under the relation (5.7), the number of the linearly independent nonzero solutions  $r'$  of the system of equations (5.4) is equal to the number of the essential and independent parameters of the group of motions, i.e. equal to the degree  $r$  of the group symmetry:

$$\mathbf{r} = \mathbf{s}(\mathbf{M} - 1)(\mathbf{M} - 2)/2 = \mathbf{m}(\mathbf{m} + \mathbf{s})/2\mathbf{s}. \quad (5.8)$$

The relations (5.7) and (5.8) among the dimensionality  $m$  of the manifold  $\mathfrak{M}$ , the rank  $M$  of the phenomenological symmetry defined on it by the function (5.1) of the binary ( $q = 2$ )  $s$ -metric geometry, and the degree  $r$  of its group symmetry are used in the definitions of §1 of this monograph, as well as in the author's monograph [10] and in his notes [11], [21], and [22]. When comparing the relations (5.7) and (5.8) with the respective relations in §1, and those in the other works mentioned, it is obviously necessary to keep in mind the following replacements:  $m \rightarrow sn$ ,  $M \rightarrow m$ .

Now let us take the ternary ( $q=3$ ) geometries.

For a ternary phenomenologically symmetric geometry of rank  $M$ , where  $M \geq 4$ , defined on one set  $\mathfrak{M}$  that is an  $m$ -dimensional manifold by the metric function (5.1), where  $\mathfrak{S}_f \subseteq \mathfrak{M}^3$ , the rank of the mapping  $F : \mathfrak{S}_F \rightarrow R^{sM(M-1)(M-2)/6}$ , where  $\mathfrak{S}_F \subseteq \mathfrak{M}^M$ , is, under Definition 1, equal to  $sM(M-1)(M-2)/6 - s$ . We shall find the rank of the mapping  $F' : \mathfrak{S}_{F'} \rightarrow R^{sM'(M'-1)(M'-2)/6}$ , where  $\mathfrak{S}_{F'} \subseteq \mathfrak{M}^{M'}$  and  $M' \geq M$ . Among all the  $sM'(M'-1)(M'-2)/6$  functions  $f|_{F'}$  of that mapping the number of the dependents is found by the method of superposition of the matrix of the triples for the cortege of length  $M$  from  $\mathfrak{S}_F$  on the matrix of the triples for the cortege of length  $M'$  from  $\mathfrak{S}_{F'}$ . The method is described above, where the binary geometries on one set are discussed. For the number of the dependent functions of the mapping  $F'$ , we similarly get the value



$s(M' - M + 1)(M' - M + 2)(M' - M + 3)/6$ . Under Definition 1, the rank of the matrix of the system of equations (5.4) will be equal to

$$\min(M'm; sM'(M' - 1)(M' - 2)/6 - s(M' - M + 1)(M' - M + 2)(M' - M + 3)/6).$$

Since  $M > 3$ , for sufficiently great values of  $M'$  that rank is equal to  $M'm$ , i.e. to the number of the unknowns in the system of equations (5.4), and it only has, for that reason, for such values of  $M'$  the zero solution. Thus, the ternary phenomenologically symmetric geometries on one set may not be endowed with a group symmetry. The similar result, and by the similar method, may be obtained for any  $q$ -ary phenomenologically symmetric geometry whose arity is more than three.

The final conclusion on the result of the above investigation is expressed in the following theorem.

**Theorem.** *Only binary ( $q = 2$ ) phenomenologically symmetric geometries may be endowed with a group symmetry of finite degree, while with the  $q$ -ary phenomenologically symmetric geometries with  $q \geq 3$  the metric function (5.1) defining them does not allow any nontrivial motions.*

This Theorem of the  $q$ -ary phenomenologically symmetric geometries with  $q \geq 3$  having no group symmetry seems to be refuted by a very simple example. Let us assign to each triple of the points  $\langle ijk \rangle$  of the coordinate plane  $(x, y)$  the oriented area of the triangle with the apices in these points:

$$S(ijk) = \frac{1}{2} \begin{vmatrix} x_i & x_j & x_k \\ y_i & y_j & y_k \\ 1 & 1 & 1 \end{vmatrix}. \quad (5.9)$$

For an arbitrary quadruple  $\langle ijkl \rangle$  of points of the plane, the four areas of the triangles  $\langle ijk \rangle$ ,  $\langle ijl \rangle$ ,  $\langle ikl \rangle$ , and  $\langle jkl \rangle$  are functionally related by the following obvious relation:

$$S(ijk) - S(ijl) + S(ikl) - S(jkl) = 0. \quad (5.10)$$

Besides, the function (5.9) allows a 5-parameter group of motions:

$$x' = ax + by + c, \quad y' = gx + hy + d, \quad (5.11)$$

where  $ah - bg = 1$ .

Thus, we seemingly come to the conclusion that the 3-point function of area (5.9) does give on the coordinate plane  $(x, y)$  a ternary geometry that is, on the one hand, phenomenologically symmetric, and on the other is endowed with a group symmetry of degree 5.

However, under Definition 1, the presence of the functional relation (5.10) is not sufficient condition for the function (5.9) to give a planar ternary phenomenologically symmetric geometry of rank 4. It is also necessary, under the same Definition, that the relation (5.10) be a generating one in the sense as follows: *any other nontrivial relation must be a corollary of that relation (5.10)*.

We shall demonstrate that that very condition the equation (5.10) does not satisfy. We shall take on a plane a heptuple of points  $< ijklpmn >$  and assign to it, according the function (5.9), all the thirty-five areas  $S(ijk), S(ijl), \dots, S(imn); S(jkl), S(jkp), \dots, S(pmn)$  of the respective triangles  $< ijk >, \dots, < pmn >$ . Out of these thirty-five areas, it is possible to eliminate, under the relation (5.10), with respect to the relation, twenty:  $S(jkl), \dots, S(pmn)$ . Among the remaining fifteen areas,  $S(ijk), S(ijl), \dots, S(imn)$  there is one trivial relation, for they depend on fourteen coordinates  $x_i, y_i, x_j, y_j, \dots, x_n, y_n$  only. If the relation (5.10) is an originating one, then, eliminating for example the area  $S(ijk)$  yields fourteen areas

$$S(ijl), S(ijp), \dots, S(imn), \quad (5.12)$$

that must be independent. There may not exist a trivial relation among them, because they are functions of the fourteen coordinates of the septuple  $< ijklpmn >$ , but there are nontrivial relations among them, which is easily made sure. If there did not exist such relations then the seven-point figure  $< ijklpmn >$  would not have had any single degree of freedom of motion, because on the fourteen coordinates of the points of it there would have been imposed as many independent correlations conditioned by the invariance of the fourteen areas (5.12). But in reality, the seven-point

figure  $\langle ijklpmn \rangle$  may move on the plane, while preserving the areas of all the thirty-five triangles, with five degrees of freedom: one rotation, two parallel translations, and two shifts. The corresponding 5-parameter group of motions is defined by the equations (5.11). The contradiction established is that what proves that among the fourteen areas (5.12) there must exist additional nontrivial functional relations that are not corollaries of the relation (5.10), and thus that last relation is not an originating one. So the function (5.9) does not give on the plane  $\mathfrak{M} = R^2$  a ternary phenomenologically symmetric geometry of rank 4 in the sense of Definition 1, despite the presence of the functional relation (5.10), because the rank of the projection of the mapping  $F' : \mathfrak{M}^7 \rightarrow R^{35}$  defined by the fourteen areas (5.12) and not including the mapping  $F : \mathfrak{M}^4 \rightarrow R^4$  is less than fourteen, i.e. it is not maximal.

The group symmetry of binary phenomenologically symmetric geometries, which are the main subject in the author's monograph "Polymetric geometries" [10], as well as in Chapter 1 of this monograph, is a determining one. That is, the function  $f : \mathfrak{S}_f \rightarrow R^s$ , where  $\mathfrak{S}_f \subseteq \mathfrak{M}^2$ , will define a phenomenologically symmetric geometry if and only if it allows a nontrivial finite-dimensional group of motions. The condition of a phenomenologically symmetric geometry being endowed with a group symmetry of finite degree determines that degree, by establishing the relation of it with the dimensionality of the manifold and the rank of the phenomenological symmetry, by the relations (5.7) and (5.8). On the other hand, without the assumption of the group symmetry in the sense of Definition 2 even the relation (5.7), that establishes the connection between the dimensionality of the manifold and the rank of the phenomenological symmetry, and having no degree of the group symmetry, must be stipulated in a complementary way without any sufficient grounding of such connection.

## §6. Functional equations in geometry

Let us consider the one-dimensional geometry that is defined by the

nondegenerate metric function

$$f(ij) = f(x_i, x_j). \quad (6.1)$$

Under Definition 1 of §1, its phenomenological symmetry of rank 3 is expressed by the equation

$$\Phi(f(ij), f(ik), f(jk)) = 0. \quad (6.2)$$

The equation (6.2) must turn into an identity in the coordinates  $x_i, x_j, x_k$  of the triple of points  $\langle ijk \rangle$  when the function (6.1) is substituted into it. Thus, the equation (6.2) is in fact the functional equation both with respect to the metric function (6.1), and as concerns the function  $\Phi$  that expresses the phenomenological symmetry of the one-dimensional geometry.

Further we shall describe in brief a method of solution of the functional equation (6.2). First, we shall set it down in the form solved with respect to one of the arguments:

$$f(ij) = \varphi(f(ik), f(jk)), \quad (6.3)$$

where  $\varphi(u, v)$  is a smooth function of 2 variables with unequal to zero derivatives  $\varphi_u$  and  $\varphi_v$ .

We take, then, an ordered quadruple of points  $\langle ijkl \rangle$  and write the equation (6.3) for the triples  $\langle ijk \rangle$ ,  $\langle ijl \rangle$ ,  $\langle ikl \rangle$ ,  $\langle jkl \rangle$ :

$$\left. \begin{aligned} f(ij) &= \varphi(f(ik), f(jk)), \\ f(ij) &= \varphi(f(il), f(jl)), \\ f(ik) &= \varphi(f(il), f(kl)), \\ f(jk) &= \varphi(f(jl), f(kl)), \end{aligned} \right\}$$

wherefrom we readily get the equality

$$\varphi[\varphi(f(il), f(kl)), \varphi(f(jl), f(kl))] = \varphi(f(il), f(jl)),$$

where, obviously, the variables  $f(il)$ ,  $f(jl)$ , and  $f(kl)$  are independent. If we introduce for them designation  $x = f(il)$ ,  $y = f(jl)$ ,  $z = f(kl)$ , we arrive at the functional equation

$$\varphi(\varphi(x, z), \varphi(y, z)) = \varphi(x, y). \quad (6.4)$$

that has a nontrivial solution as follows:

$$\varphi(u, v) = \psi(\psi^{-1}(u) - \psi^{-1}(v)), \quad (6.5)$$

where  $\psi$  is an arbitrary smooth function of one variable with  $\psi' \neq 0$ , and  $\psi^{-1}$  is the function that is its inverse.

By using the solution (6.5) of the equation (6.3), we arrive at the equation (6.2):

$$\psi^{-1}(f(ij)) - \psi^{-1}(f(ik)) + \psi^{-1}(f(jk)) = 0. \quad (6.6)$$

The explicit form of the metric function (6.1) itself can again be found from the equation (6.3) with the solution (6.5), if we fix in it the coordinate  $x_k$  of the point  $k$ :

$$f(ij) = \psi(\varphi(x_i) - \varphi(x_j)), \quad (6.7)$$

where  $\varphi(x) = \psi^{-1}(f(x, x_k))|_{x_k=const}$ .

Together, the equation (6.6) and the function (6.7) are a general solution of the functional equation (6.2). With an accuracy up to a change of coordinates  $\varphi(x) \rightarrow x$  in the one-dimensional manifold and a scaling transformation  $\psi^{-1}(f) \rightarrow f$  of the metric function, that solution may be written in the following canonical form:

$$\left. \begin{aligned} f(ij) &= x_i - x_j, \\ f(ij) - f(ik) + f(jk) &= 0. \end{aligned} \right\} \quad (6.8)$$

A smooth invertible transformation of the one-dimensional manifold

$$x' = \lambda(x) \quad (6.9)$$

is called a motion if it preserves the metric function:  $f(i'j') = f(ij)$ . Hence, with the known metric function (6.1), we get the functional equation for the set of motions:

$$f(\lambda(x_i), \lambda(x_j)) = f(x_i, x_j), \quad (6.10)$$

whose solution for a one-dimensional phenomenologically symmetric geometry is the 1-parameter group

$$x' = \lambda(x; a). \quad (6.11)$$

And vice versa, if the one-parameter group of transformations (6.11) is known the metric function of the one-dimensional geometry for which that group is the group of motions will be found as its two-point invariant, through the solution of the functional equation as follows

$$f(\lambda(x_i; a), \lambda(x_j; a)) = f(x_i, x_j). \quad (6.12)$$

Let the infinitesimal operator

$$X = \lambda(x)\partial/\partial x \quad (6.13)$$

belongs to the Lie algebra of the group of motions (6.11). Then the metric function is simultaneously the solution of the differential equation

$$X(i)f(ij) + X(j)f(ij) = 0, \quad (6.14)$$

which, in its turn, and with the metric function (6.1) known, is the functional equation for the coefficient  $\lambda(x)$  of the operator (6.13).

The two-dimensional geometry is defined by the nondegenerate metric function

$$f(ij) = f(x_i, y_i, x_j, y_j), \quad (6.15)$$

and its phenomenological symmetry is expressed by the equation

$$\Phi(f(ij), f(ik), f(il), f(jk), f(jl), f(kl)) = 0. \quad (6.16)$$

If the metric function (6.15) is substituted into the equation (6.16) then, with respect to the eight coordinates  $x_i, y_i, x_j, y_j, x_k, y_k, x_l, y_l$  of the points of the quadruple  $\langle ijkl \rangle$ , it must turn into an identity. Thus, that equation is really a special kind of a functional equation both with respect of the metric function  $f$  and with respect of the function  $\Phi$ , which is part of the phenomenological symmetry of the two-dimensional geometry. With an accuracy up to a change of coordinates in a two-dimensional manifold and a scaling transformation  $\psi(f) \rightarrow f$ , all the possible solutions of the equation (6.16), with respect of the metric function (6.15), may be written in the eleven canonical forms (2.7) – (2.17). As to the function  $\Phi$ , it cannot always be written in the explicit form, of which we spoke in more detail in §2, after the said classification had been given.

The set of invertible motions

$$x' = \lambda(x, y), \quad y' = \sigma(x, y), \quad (6.17)$$

that preserve the metric function (6.15) is the totality of the solutions of the functional equation

$$f(\lambda(i), \sigma(i), \lambda(j), \sigma(j)) = f(x_i, y_i, x_j, y_j). \quad (6.18)$$

Under Theorem 3 of §1, that totality is the three-parameter group of transformations of the manifold:

$$x' = \lambda(x, y; a^1, a^2, a^3), \quad y' = \sigma(x, y; a^1, a^2, a^3), \quad (6.19)$$

and defines the group symmetry of the corresponding two-dimensional geometry. And if the group of transformations (6.19) is known, then, with an accuracy up to a scaling transformation, the metric function (6.15) is reconstructed as its nondegenerate two-point invariant by way of the solution of another functional equation:

$$f(\lambda(i; a), \sigma(i; a), \lambda(j; a), \sigma(j; a)) = f(x_i, y_i, x_j, y_j), \quad (6.20)$$

where, for example,  $\lambda(i; a) = \lambda(x_i, y_i; a^1, a^2, a^3)$ .

Let

$$X = \lambda(x, y) \partial / \partial x + \sigma(x, y) \partial / \partial y \quad (6.21)$$

be an infinitesimal operator of the three-dimensional Lie algebra of the group of motions (6.19). Then the metric function (6.15) is also a solution of the differential equation

$$X(i)f(ij) + X(j)f(ij) = 0. \quad (6.22)$$

However, with the metric function (6.15) known, it is already the functional equation for the coefficients  $\lambda$  and  $\sigma$  of the operator (6.21).

We shall call *a cycle of a two-dimensional geometry* such a smooth nondegenerate curve

$$x = x(t), \quad y = y(t), \quad (6.23)$$

along which a rigid triangle  $\langle ijk \rangle$  may roll freely. It is obvious that on the set of the points of that curve the metric function (6.15) must give a phenomenologically symmetric one-dimensional geometry endowed with a group symmetry of degree 1. As result, we have the functional equation for the cycle [23, §12] as follows:

$$f(x(t_i), y(t_i), x(t_j), y(t_j)) = \psi(t_i - t_j). \quad (6.24)$$

For example, for the Euclidean plane the functional equation

$$(x(t_i) - x(t_j))^2 + (y(t_i) - y(t_j))^2 = \psi(t_i - t_j)$$

has two solutions [23, §15]:

$$1) \ x = at + b, \ y = ct + d; \quad 2) \ x = R \cos t + x_0, \ y = R \sin t + y_0,$$

where  $a^2 + c^2 \neq 0$ ,  $R > 0$  giving on the plane a set of straight lines and a set of circles.

We shall note that functional equations similar to the equations (6.18), (6.20), (6.22), (6.24) may be written for any geometry defined by the metric function (1.2). For example, for the three-dimensional geometry defined by the metric function (2.20), the functional equation for the cycle will be as follows:

$$f(x(t_i), y(t_i), z(t_i), x(t_j), y(t_j), z(t_j)) = \psi(t_i - t_j),$$

and for the three-dimensional geometry defined by the metric function (3.14), the system of functional equations for the cycle will be written in the same way, but in it  $f = (f^1, f^2, f^3)$  and  $\psi = (\psi^1, \psi^2, \psi^3)$ . The method of solution for most geometrical functional equations is that of reduction to differential ones and of separating the variables.

## **§7. Problems of classification of phenomenologically symmetric geometries**

In the theory of physical structures, classification of phenomenologically symmetric geometries is one of the most important problems. The thing is, both the metric functions (1.2) themselves, defining such geometries, and the equations (1.1), expressing their phenomenological symmetry, may have an essential physical interpretation.

In §2 and §3 the complete classifications are given that have been built of one-dimensional, two-dimensional, and three-dimensional phenomenologically symmetric geometries of respective ranks 3, 4, and 5, as well as of the dimetric, trimetric and four-metric phenomenologically symmetric geometries of the minimal rank, i.e. that equal to 3. Any other classifications have not yet been built, because up till now we have not found any new methods of solving problems of that kind.



Now, we shall give the classification problems that are, on the one hand, a natural extension of those already solved, and, on the other, may interest those readers who will be able to find more effective methods of their solution.

### 1. *Four-dimensional geometries*

A four-dimensional geometry is defined on a four-dimensional manifold  $\mathfrak{M}$  by the nondegenerate metric function

$$f(ij) = f(x_i, y_i, z_i, t_i, x_j, y_j, z_j, t_j), \quad (7.1)$$

and its phenomenological symmetry is expressed by the equation

$$\Phi(f(ij), f(ik), f(il), f(im), f(in), f(jk), f(jl), f(jm), f(jn), \dots, f(mn)) = 0, \quad (7.2)$$

that establishes the relation of 15 reciprocal distances among the six points of a cortege  $\langle ijklmn \rangle$  of some set open and dense in  $\mathfrak{M}^6$ . The group symmetry such a geometry is endowed with is of degree 10.

A preliminary classification of four-dimensional geometries was given in the end of §2. That classification cannot be considered complete, for the methods employed in building it cannot help overcome technical difficulties that have been encountered. Recently, V.A. Kyrov has been developing a new method of their classification, one based on the hypothesis of enclosure. The essence of the hypotheses is illustrated by the classification (2.23) – (2.37) of the three-dimensional phenomenologically symmetric geometries where it can be seen that each metric function of it contains in itself as a whole a metric function defining a two-dimensional phenomenologically symmetric geometry:

$$f(ij) = f(g(ij), z_i, z_j),$$

where the expression for  $g(ij) = g(x_i, y_i, x_j, y_j)$  is borrowed from the classification (2.7) – (2.17). Unfortunately, we have not any rigorous proof of the hypothesis of enclosure, but it seems to be confirmed by those complete classifications that are already available.

### 2. *Dimetric and trimetric geometries*

The dimetric and trimetric phenomenologically symmetric geometries of minimal rank 3 were discussed in §3, and complete classifications (3.12) - (3.13) and (3.28) - (3.38) have been built for them. So it is natural to undertake next classifying dimetric and trimetric phenomenologically symmetric geometries of higher rank, that is of rank 4. For example, the dimetric phenomenologically symmetric geometry of rank 4 is defined on a four-dimensional manifold by the two-component metric function

$$f(ij) = (f^1(ij), f^2(ij)) = f(x_i, y_i, z_i, t_i, x_j, y_j, z_j, t_j), \quad (7.3),$$

and its phenomenological symmetry is expressed by the equation

$$\Phi(f(ij), f(ik), f(il), f(jk), f(jl), f(kl)) = 0, \quad (7.4)$$

where  $\Phi$  is a two-component function of 12 variables. The degree of its group symmetry is 6.

The complete classification of such geometries naturally includes all the pairs of the metric functions of the classification (2.7) – (2.17), with  $f^1(ij) = f^1(x_i, y_i, x_j, y_j)$ ,  $f^2(ij) = f^2(z_i, t_i, z_j, t_j)$ . For example, the combination of the functions (2.7) and (2.12), giving Euclidean plane and the symplectic plane:

$$f^1(ij) = (x_i - x_j)^2 + (y_i - y_j)^2, \quad f^2(ij) = z_i t_j - z_j t_i$$

is a two-component metric function that defines one of the dimetric four-dimensional phenomenologically symmetric geometries of rank 4. The complete number of combinations, including the diagonal ones, is 65. But it is possible there exist such dimetric geometries whose metric functions (7.3) are not such combinations.

### 3. Polymetric geometries

The classifications of some phenomenologically symmetric polymetric geometries of minimal rank, equal to 3, were given in §3. Such geometries are defined in the space  $R^s$  by an  $s$ -component metric function

$$f(ij) = (f^1(ij), \dots, f^s(ij)) = f(x_i^1, \dots, x_i^s, x_j^1, \dots, x_j^s),$$

and their phenomenological symmetry is expressed by the equation

$$\Phi(f(ij), f(ik), f(jk)) = 0,$$

where  $\Phi$  is an  $s$ -component function of  $3s$  variables. The complete classification has only been built for  $s = 1, 2, 3$ , and 4 and is not available for  $s \geq 5$ .

Thus, the classification of the phenomenologically symmetric geometries is not finished yet. So it makes sense to present in a visually graspable form the problems both solved and not yet solved. Then, anyone who would like to try one's abilities can choose any of them (the solved included, for developing new methods of classification and inspecting the results obtained). We shall represent the list of problems in a table format as follows:

<b>A classification of phenomenologically symmetric geometries</b>							
$\text{№}$	$s$	$n$	$sn$	$m = n + 2$	$sn(n + 1)/2$	solved	source
1	1	1	1	3	1	+	§2
2	1	2	2	4	3	+	§2
3	1	3	3	5	6	+	§2
4	1	4	4	6	10	—	—
5	1	$\geq 5$	$5, 6, \dots$	$7, 8, \dots$	$15, 21, \dots$	—	—
6	2	1	2	3	2	+	§3
7	2	$\geq 2$	$4, 6, \dots$	$4, 5, \dots$	$6, 12, \dots$	—	—
8	3	1	3	3	3	+	§3
9	3	$\geq 2$	$6, 9, \dots$	$4, 5, \dots$	$9, 18, \dots$	—	—
10	4	1	4	3	4	+	§3
11	4	$\geq 2$	$8, 12, \dots$	$4, 5, \dots$	$12, 24, \dots$	—	—
12	$\geq 5$	$\geq 1$	$\geq 5$	$\geq 3$	$\geq 5$	—	—

We shall remind that  $s$  is the number of the components of the metric function  $f = (f^1, \dots, f^s)$  that defines on an  $sn$ -dimensional manifold a phenomenologically symmetric geometry of rank  $m = n + 2$  whose group of motions depends on  $sn(n + 1)/2$  continuous parameters. In the two right-hand columns the plus sign means that the problem has been solved and the number of the paragraph is given where the corresponding complete classification is to be found.

The metric functions defining on the manifold phenomenologically symmetric geometries are nondegenerate two-point invariants of some groups of transforma-

tions of the manifold. The problem of their classification implied building up a preliminary classification of the groups of transformations with the certain number of the continuous parameters. But, with the number of the components of the metric function and the rank of the phenomenological symmetry of the geometry it defines increasing, classifying groups of motions becomes rather a strenuous task, in the technical sense. So, quite naturally a question suggests itself: Is a preliminary classification of the groups of transformations of a manifold really necessary. After all, for many of them the two-point invariants turn out to be degenerate. Therefore, it seems to make sense first to establish what condition must a group of transformations satisfy in order that the two-point invariant of it be nondegenerate. For example, in building the classification (3.28)–(3.38) of the three-dimensional trimetric phenomenologically symmetric geometries of rank 3 such condition was that of transitivity of the group of transformations. But at large the problem of additional constraints on groups of transformations following from their two-point invariants being nondegenerate is still open. We shall also note that all the definitions and results of Chapter 1 are local. Their globalization requires a new qualitative approach in the development of the methods of research and presents a new problem, whose significance is conditioned by the possibility of essential interpretations of the phenomenologically symmetric geometries, not only in geometry, but in physics too.

## CHAPTER II

### A physical structure as a geometry of two sets

#### §8. The phenomenological and group symmetry of physical structures

Let there be two sets  $\mathfrak{M}$  and  $\mathfrak{N}$  that are an  $sm$ -dimensional and an  $sn$ -dimensional manifolds, where  $s, m$  and  $n$  are natural numbers, whose points we shall designate by Latin and Greek lower-case letters respectively, and a function  $f : \mathfrak{M} \times \mathfrak{N} \rightarrow R^s$  that assigns to a pair  $\langle i\alpha \rangle$  from its domain  $\mathfrak{S}_f \subseteq \mathfrak{M} \times \mathfrak{N}$  some collection of  $s$  real numbers  $f(i\alpha) = (f^1(i\alpha), \dots, f^s(i\alpha)) \in R^s$ . We shall note that in the general case  $\mathfrak{S}_f \neq \mathfrak{M} \times \mathfrak{N}$ , i.e. the function  $f$  does not assign  $s$  numbers to *every* pair from  $\mathfrak{M} \times \mathfrak{N}$  but in the further discussion it will be convenient in the explicit writing of the value  $f(i\alpha)$  of the function for a pair  $\langle i\alpha \rangle$  to consider that  $\langle i\alpha \rangle \in \mathfrak{S}_f$ . We shall designate by  $U(i)$  and  $U(\alpha)$  the neighbourhoods of the points  $i \in \mathfrak{M}$  and  $\alpha \in \mathfrak{N}$  and by  $U(\langle i\alpha \rangle)$  the neighbourhood of the pair  $\langle i\alpha \rangle \in \mathfrak{M} \times \mathfrak{N}$ , and similarly the neighbourhoods of the corteges from the other direct products of the sets  $\mathfrak{M}$  and  $\mathfrak{N}$  each by itself or by each other.

For some corteges  $\langle \alpha_1 \dots \alpha_m \rangle \in \mathfrak{N}^m$  and  $\langle i_1 \dots i_n \rangle \in \mathfrak{M}^n$ , we shall introduce a function  $f^m = f[\alpha_1 \dots \alpha_m]$  and  $f^n = f[i_1 \dots i_n]$ , by assigning to the points  $i \in \mathfrak{M}$  and  $\alpha \in \mathfrak{N}$  points  $(f(i\alpha_1), \dots, f(i\alpha_m)) \in R^{sm}$  and  $(f(i_1\alpha), \dots, f(i_n\alpha)) \in R^{sn}$  if the pairs  $\langle i\alpha_1 \rangle, \dots, \langle i\alpha_m \rangle$  and  $\langle i_1\alpha \rangle, \dots, \langle i_n\alpha \rangle$  belong to  $\mathfrak{S}_f$ . We shall note that the functions  $f^m$  and  $f^n$  are not necessarily defined everywhere on the sets  $\mathfrak{M}$  and  $\mathfrak{N}$ . We shall assume that three axioms hold as follows:

**I.** The domain  $\mathfrak{S}_f$  of the function  $f$  is a set that is open and dense in  $\mathfrak{M} \times \mathfrak{N}$ .

**II.** The function  $f$  in its domain is sufficiently smooth.

**III.** In  $\mathfrak{N}^m$  and  $\mathfrak{M}^n$  the sets of corteges of lengths  $m$  and  $n$  are dense such for which the functions  $f^m$  and  $f^n$  have maximal ranks equal to  $sm$  and  $sn$  in the points of sets dense in  $\mathfrak{M}$  and  $\mathfrak{N}$  respectively.

Sufficient smoothness means that both the function  $f$  and all its derivatives of sufficiently high order are continuous in the domain of the function  $f$ . We shall call a smooth function  $f$  that satisfies Condition III a *nondegenerate* one. We shall also note that the restriction in Axioms I, II, and III by open and dense subsets is due to the possibility that the original sets may contain exceptional subsets of smaller dimensionality where those axioms do not hold.

We shall also introduce a function  $F$ , by assigning to a cortege  $\langle ijk \dots v, \alpha\beta\gamma \dots \tau \rangle$  of length  $m+n+2$  from  $\mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1}$  the point  $(f(i\alpha), f(i\beta), \dots, f(v\tau)) \in R^{s(m+1)(n+1)}$  whose coordinates in  $R^{s(m+1)(n+1)}$  are determined by the series of values of the function  $f$  for all the pairs of the elements of the original cortege  $(\langle i\alpha \rangle, \langle i\beta \rangle, \dots, \langle v\tau \rangle)$  ordered by that original cortege if all those pairs belong to  $\mathfrak{S}_f$ . The domain of the function introduced is, obviously, a set open and dense in  $\mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1}$ . We shall designate it by  $\mathfrak{S}_F$ .

**Definition 1.** We shall say that the function  $f = (f^1, \dots, f^s)$  gives on an  $sm$ -dimensional and an  $sn$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a *physical structure (a phenomenologically symmetric geometry of two sets) of rank  $(n+1, m+1)$*  if, in addition to Axioms I, II, and III, one more axiom holds as follows:

**IY.** There exists a set dense in  $\mathfrak{S}_F$  for whose every cortege  $\langle ijk \dots v, \alpha\beta\gamma \dots \tau \rangle$  of length  $m+n+2$  and some neighbourhood  $U(\langle i \dots \tau \rangle)$  of it a sufficiently smooth  $s$ -component function  $\Phi : \mathcal{E} \rightarrow R^s$  may be found defined in some region  $\mathcal{E} \subset R^{s(m+1)(n+1)}$  that contains the point  $F(\langle i \dots \tau \rangle)$ , such that  $\text{rang } \Phi = s$  and the set  $F(U(\langle i \dots \tau \rangle))$  is a subset of the set of zeros of the function  $\Phi$ , that is

$$\Phi(f(i\alpha), f(i\beta), \dots, f(v\tau)) = 0 \quad (8.1)$$

for all the corteges from  $U(\langle ijk \dots v, \alpha\beta\gamma \dots \tau \rangle)$ .

Axiom IY gives the essence of the principle of phenomenological symmetry. The equations (8.1) define  $s$  functional relations among  $s(m+1)(n+1)$

values of  $s$  physical quantities  $f = (f^1, \dots, f^s)$  measured by experiment and are an analytical expression of a law of physics written in the phenomenologically symmetric form. The condition of rang  $\Phi = s$  means that the equations  $\Phi = 0$  (i.e.  $\Phi_1 = 0, \dots, \Phi_s = 0$ ) are independent.

Let  $x = (x^1, \dots, x^{sm})$  and  $\xi = (\xi^1, \dots, \xi^{sn})$  be local coordinates in the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ . Then for the original function  $f$ , we have, in some neighbourhood  $U(i) \times U(\alpha)$  of every pair  $\langle i\alpha \rangle \in \mathfrak{S}_f$ , the local coordinate representation

$$f(i\alpha) = f(x_i, \xi_\alpha) = f(x_i^1, \dots, x_i^{sm}, \xi_\alpha^1, \dots, \xi_\alpha^{sn}), \quad (8.2)$$

whose properties are determined by Axioms II and III. Since under Axiom III the ranks of the functions  $f^m$  and  $f^n$  are maximal, the coordinates  $x$  and  $\xi$  are included in the representation (8.2) in an essential manner. The latter implies that no smooth local invertible change of coordinates may result in their number in the representation (8.2) being decreased, i.e. it may not, in any local system of coordinates, be written as

$$f(i\alpha) = f(x_i^1, \dots, x_i^{m'}, \xi_\alpha^1, \dots, \xi_\alpha^{n'}),$$

where either  $m' < sm$ , or  $n' < sn$ . Indeed, if for example  $m' < sm$ , then for any cortege  $\langle \alpha_1 \dots \alpha_m \rangle \in (U(\alpha))^m$  of length  $m$  and for any point of  $U(i)$  the rank of the function  $f^m = f[\alpha_1 \dots \alpha_m]$  will a fortiori be less than  $sm$ , which is in contradiction with Axiom III. We shall note, however, that the essential dependence of the representation (8.2) on the local coordinates  $x_i$  and  $\xi_\alpha$  is not a sufficient guaranty of Axiom III being satisfied. That is, the function  $f$  may turn out to be degenerate, with all the coordinates present in any such representation.

We shall consider the function  $f = (f^1, \dots, f^s)$  as a metric one in a geometry of two sets. But since  $s$  distances  $f(i\alpha)$  are determined for the points of more than one set, the ordinary metric axioms are of no sense here.

Using the representation (8.2), let us write the local coordinate definition for the function  $F$  that we have introduced:

$$\left. \begin{array}{l} f(i\alpha) = f(x_i, \xi_\alpha), \\ f(i\beta) = f(x_i, \xi_\beta), \\ \dots\dots\dots \\ f(v\tau) = f(x_v, \xi_\tau), \end{array} \right\} \quad (8.3)$$

whose functional matrix

$$\left\| \begin{array}{cccccc} \frac{\partial f(i\alpha)}{\partial x_i} & 0 & \dots & 0 & 0 & \frac{\partial f(i\alpha)}{\partial \xi_\alpha} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \frac{\partial f(v\tau)}{\partial x_\tau} & 0 & 0 & \dots & 0 & \frac{\partial f(v\tau)}{\partial \xi_\tau} \end{array} \right\| \quad (8.4)$$

has  $s(m+1)(n+1)$  rows and  $s(2mn+m+n)$  columns. Here, by  $\partial f/\partial x$  and  $\partial f/\partial \xi$  the respective functional matrices are briefly designated for the components of the function  $f = (f^1, \dots, f^s)$  with respect to the coordinates  $x = (x^1, \dots, x^{sm})$  and  $\xi = (\xi^1, \dots, \xi^{sn})$  respectively:

$$\begin{aligned} \partial f / \partial x &= \left\| \begin{array}{ccc} \partial f^1 / \partial x^1 & \cdots & \partial f^1 / \partial x^{sm} \\ \cdots & \cdots & \cdots \\ \partial f^s / \partial x^1 & \cdots & \partial f^s / \partial x^{sm} \end{array} \right\|, \\ \partial f / \partial \xi &= \left\| \begin{array}{ccc} \partial f^1 / \partial \xi^1 & \cdots & \partial f^1 / \partial \xi^{sn} \\ \cdots & \cdots & \cdots \\ \partial f^s / \partial \xi^1 & \cdots & \partial f^s / \partial \xi^{sn} \end{array} \right\|. \end{aligned}$$

The definition (8.3) for the function  $F$  is a system of  $s(m+1)(n+1)$  functions  $f^1(i\alpha), \dots, f^s(i\alpha), \dots, f^1(v\tau), \dots, f^s(v\tau)$  that depend in a special manner on  $s(2mn+m+n)$  variables  $x_i^1, \dots, x_i^{sm}, \dots, \xi_\tau^1, \dots, \xi_\tau^{sn}$  – the coordinates of all the points of the cortege  $\langle ijk\dots v, \alpha\beta\gamma\dots\tau \rangle$  of length  $m+n+2$ . Since the number of functions in the system (8.3) is not more than the total number of the variables, the presence of the relation (8.1) is a nontrivial fact, not taking place for arbitrary functions in that system.

The geometries of one set discussed in Chapter I demonstrate that their phenomenological and group symmetry mutually condition each other. Thus, the relation among the six distances for any four points of a two-dimensional geometry, not only the Euclidean one, results in the appearing of a three-parameter group of motions in it. But motion in a geometry of two sets is



specific, and very different from that in a geometry of one set. That is why it is pertinent that exact definitions should be given.

Under the name of a local motion in a geometry of two sets  $\mathfrak{M}$  and  $\mathfrak{N}$  we shall understand such a pair of biunique (one-to-one) smooth mappings (transformations)

$$\lambda : U \rightarrow U' \text{ и } \sigma : V \rightarrow V', \quad (8.5)$$

where  $U, U' \subset \mathfrak{M}$  and  $V, V' \subset \mathfrak{N}$  are open regions, at which the function  $f$  is preserved. The latter means that for every pair  $\langle i\alpha \rangle \in \mathfrak{S}_f$ , such that  $i \in U$ ,  $\alpha \in V$  and  $\langle i'\alpha' \rangle \in \mathfrak{S}_f$ , where  $i' = \lambda(i) \in U'$ ,  $\alpha' = \sigma(\alpha) \in V'$ , the equality

$$f(\lambda(i), \sigma(\alpha)) = f(i\alpha), \quad (8.6)$$

takes place that is satisfied for each component  $f^1, \dots, f^s$  of the function  $f$ .

The set of all motions (8.5) is a local group of transformations for which the function  $f$ , under the equality (8.6), is a *two-point invariant*. The transformations  $\lambda$  and  $\sigma$  in the motions (8.5) are themselves two separate groups, and the group of motions is their mutual extension. If the function  $f$  is known, in its local coordinate representation (8.2), for example, then the equality (8.6) is the functional equation with respect to the transformations  $\lambda$  and  $\sigma$ . However, we only know about the function  $f$  that it is nondegenerate and satisfies some system of  $s$  independent equations (8.1). But that turns out to be enough to establish the fact of existence of the group of motions of it depending on  $smn$  parameters.

**Definition 2.** We shall say that the function  $f = (f^1, \dots, f^s)$  gives on an  $sm$ -dimensional and an  $sn$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a *geometry of two sets endowed with a group symmetry of degree  $smn$*  if in addition to Axioms I, II, and III, one more axiom holds as follows:

**IY'.** There exist sets open and dense in  $\mathfrak{M}$  and  $\mathfrak{N}$  for all the points  $i$  and  $\alpha$  of which effective smooth actions of an  $smn$ -dimensional local Lie group are defined in some neighbourhoods  $U(i)$  and  $U(\alpha)$ , such that its actions in the neighbourhoods  $U(i), U(j)$  and  $U(\alpha), U(\beta)$  of the points  $i, j$  and  $\alpha, \beta$  coincide in the intersections  $U(i) \cap U(j)$  and  $U(\alpha) \cap U(\beta)$  and the function  $f$  is a two-point invariant in each of its  $s$  components.

We shall remind that groups of Lie transformations of smooth manifolds were described in §1 when a similar axiom, Axiom IV' was formulated of a geometry of one set. The groups of transformations in question in Axiom IY' of this paragraph define a peculiar local mobility of rigid figures ("solid bodies") in the space  $\mathfrak{M} \times \mathfrak{N}$  with  $smn$  degrees of freedom. We shall note that global mobility is not necessarily implied. The set of pairs  $\langle i\alpha \rangle$  for which the function  $f$  is defined and is simultaneously the two-point invariant is, obviously, open and dense in  $\mathfrak{M} \times \mathfrak{N}$ .

Under Axiom IY', there are  $smn$ -dimensional linear families of smooth vector fields  $X$  and  $\Xi$  defined on sets open and dense in  $\mathfrak{M}$  and  $\mathfrak{N}$  that are commutation closed, i.e. algebras of Lie transformations. We shall write the basic vector fields of these families, in some local systems of coordinates in the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ , in the operator form:

$$\left. \begin{aligned} X_\omega &= \lambda_\omega^\mu(x) \partial / \partial x^\mu, \\ \Xi_\omega &= \sigma_\omega^\nu(\xi) \partial / \partial \xi^\nu, \end{aligned} \right\} \quad (8.7)$$

where  $\omega = 1, \dots, smn$ , and with respect to "mute" indexes  $\mu$  and  $\nu$  summation is performed from 1 to  $sm$  and from 1 to  $sn$  respectively. By the criterion of invariance, the function  $f(i\alpha)$  will be the invariant of the local group of transformations of some neighbourhood  $U(i) \times U(\alpha)$ , i.e. a two-point invariant, if and only if it satisfies component-wise a system of  $smn$  equations

$$X_\omega(i)f(i\alpha) + \Xi_\omega(\alpha)f(i\alpha) = 0 \quad (8.8)$$

with the operators (8.7):

$$\lambda_\omega^\mu(i) \frac{\partial f(i\alpha)}{\partial x_i^\mu} + \sigma_\omega^\nu(\alpha) \frac{\partial f(i\alpha)}{\partial \xi_\alpha^\nu} = 0, \quad (8.9)$$

where  $\lambda_\omega^\mu(i) = \lambda_\omega^\mu(x_i) = \lambda_\omega^\mu(x_i^1, \dots, x_i^{sm})$ ,  $\sigma_\omega^\nu(\alpha) = \sigma_\omega^\nu(\xi_\alpha) = \sigma_\omega^\nu(\xi_\alpha^1, \dots, \xi_\alpha^{sn})$ .

**Theorem 1.** *If a function  $f = (f^1, \dots, f^s)$  gives on an  $sm$ -dimensional and an  $sn$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a geometry of two sets endowed with a group symmetry of degree  $smn$  then, on the same manifolds, it gives a physical structure (a phenomenologically symmetric geometry of two sets) of rank  $(n + 1, m + 1)$ .*

**Theorem 2.** *If a function  $f = (f^1, \dots, f^s)$  gives on an  $sm$ -dimensional*

and an  $sn$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a physical structure (a phenomenologically symmetric geometry of two sets) of rank  $(n+1, m+1)$  then, on the same manifolds, it gives a geometry of two sets endowed with a group symmetry of degree  $smn$ .

The complete proofs of these theorems, that are each the inverse of the other, may be found in §1 of the author's monograph [24]. Their corollary is the conclusion about the phenomenological and the group symmetries of a geometry of two sets defined on an  $sm$ -dimensional and an  $sn$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  by the  $s$ -component metric function  $f = (f^1, \dots, f^s)$  being equivalent.

**Theorem 3.** *For a function  $f = (f^1, \dots, f^s)$  to define on an  $sm$ -dimensional and an  $sn$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a geometry of two sets endowed with a group symmetry of degree  $smn$  it is necessary and sufficient that it should give, on the same manifolds, a physical structure (a phenomenologically symmetric geometry of two sets) of rank  $(n+1, m+1)$ .*

In §9 a complete classification of unimetric ( $s = 1$ ) physical structures of arbitrary rank  $(n+1, m+1)$  will be given. We shall note that for  $s \geq 2$  a complete classification of polymetric physical structures of rank  $(n+1, m+1)$  has not been built yet. However, with respect to those functions too some preliminary results have been obtained. In particular, the equivalence of the phenomenological and group symmetries established by Theorem 3 was used by the author [25], [26] in building the classification of the dimetric physical structures of rank  $(n+1, 2)$ , i.e. for the case of  $s = 2$  and  $m = 1$ ,  $n \geq 1$ . That classification is given in §10 of this monograph. Besides, since trimetric physical structures of rank  $(2, 2)$  allow three-dimensional groups of motions, it turned out to be possible to use the classification available in [17] of three-dimensional Lie algebras of transitive transformations of the space to build the classification of the physical structures of rank  $(2, 2)$  too (given in §10 of this monograph and in the monograph [24]).

Of special interest are complex physical structures. For example, Yu.S. Vladimirov [27] used a complex structure of rank  $(3, 3)$  in a theoretical justification of the dimensionality and signature of the classical space-time.

He used complex physical structures of higher rank for building a unified theory of physical interactions.

Mathematically, the complex physical structures are a special case of real dimetric physical structures. So, if a complete classification of the latter should be built one of the former could be reproduced after it. Also, complex physical structures may be derived from real unimetric ones, by way of their complexification which consists in replacing of the real coordinates and functions by complex ones. But that method does not carry with itself any guaranty of a classification of the complex physical structures expected to be quite complete.

### §9. A classification of unimetric physical structures

Under §8, a unimetric phenomenologically symmetric geometry of two sets (a physical structure) of rank  $(n+1, m+1)$  may, grosso modo, be defined as follows. Let the sets  $\mathfrak{M}$  and  $\mathfrak{N}$  be respectively an  $m$ -dimensional and an  $n$ -dimensional smooth manifolds. We shall designate the local coordinates of these manifolds by  $x = (x^1, \dots, x^m)$  and  $\xi = (\xi^1, \dots, \xi^n)$  considering, for the sake of definiteness, that  $m \leq n$ . Let, further, there exist a function  $f : \mathfrak{M} \times \mathfrak{N} \rightarrow R$  with the domain open and dense in  $\mathfrak{M} \times \mathfrak{N}$  that assigns some number to every pair from it. We shall call the function  $f$  a metric one, and we shall not require that it should satisfy the usual metric axioms, especially as because the distances for two points from only  $\mathfrak{M}$ , or for two points from only  $\mathfrak{N}$ , are not defined. It is assumed that its local coordinate representation is defined by a sufficiently smooth function (8.2), which is more convenient to write down without specifying the designations for the points  $i$  and  $\alpha$ :

$$f = f(x, \xi) = f(x^1, \dots, x^m, \xi^1, \dots, \xi^n). \quad (9.1)$$

In consequence of the metric function  $f$  being nondegenerate, the coordinates  $x$  and  $\xi$  are included into the representation (9.1) in an essential manner. The latter means that no smooth locally invertible change of coordinates

will result in their number in the representation (9.1) being decreased.

We shall construct a function  $F : \mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1} \rightarrow R^{(n+1)(m+1)}$  with the domain natural in  $\mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1}$ , by assigning to every cortege of length  $m+n+2$  of it all  $(m+1)(n+1)$  distances possible with respect to the metric function  $f$  and ordered with respect to the aforesaid manifold. We shall say that the function  $f$  defines on an  $m$ -dimensional and an  $n$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a phenomenologically symmetric geometry of two sets (a physical structure) of rank  $(n+1, m+1)$ , if locally the set of values of the constructed function  $F$  in  $R^{(m+1)(n+1)}$  is a subset of the set of zeros of some sufficiently smooth function  $\Phi$  of  $(m+1)(n+1)$  variables with  $\text{grad}\Phi \neq 0$  on a subset dense in the domain of the function  $F$ , satisfying the equation (8.1).

In the author's note [28] the complete classification of unimetric physical structures of arbitrary rank  $(n+1, m+1)$  is given, with the natural supposition that  $n \geq m \geq 1$ , as the opposite case, that of  $m \geq n \geq 1$  is easily reproduced by the symmetry, and in his notes [29] and [30] and his monograph [31] mathematical methods are given that were used to build it.

*Further we shall write the corresponding classification results with an accuracy up to a locally invertible change of coordinates in the manifolds  $\mathfrak{M}, \mathfrak{N}$  and to the scaling transformation  $\chi(f) \rightarrow f$ , where  $\chi$  is an arbitrary smooth function of one variable with the derivative unequal to zero.*

$$m = 1, n = 1:$$

$$f = x + \xi; \tag{9.2}$$

$$m = 1, n = 2:$$

$$f = x\xi + \eta; \tag{9.3}$$

$$m = 1, n = 3:$$

$$f = (x\xi + \eta)/(x + \vartheta); \tag{9.4}$$

$$m = n \geq 2:$$

$$f = x^1\xi^1 + \dots + x^{m-1}\xi^{m-1} + x^m\xi^m, \tag{9.5}$$

$$f = x^1\xi^1 + \dots + x^{m-1}\xi^{m-1} + x^m + \xi^m; \tag{9.6}$$

$$m = n - 1 \geq 2:$$

$$f = x^1 \xi^1 + \dots + x^m \xi^m + \xi^{m+1}. \quad (9.7)$$

For all the other pairs of values of the natural numbers  $m$  and  $n$ , satisfying the above-said condition of  $n \geq m \geq 1$ , no physical structures of rank  $(n + 1, m + 1)$  exist.

The phenomenological symmetry of the geometries of two sets (physical structures) that are defined by the metric functions (9.2)–(9.7) above is, correspondingly, naturally expressed by the following equations:

$$f(i\alpha) - f(i\beta) - f(j\alpha) + f(j\beta) = 0; \quad (9.2')$$

$$\begin{vmatrix} f(i\alpha) & f(i\beta) & 1 \\ f(j\alpha) & f(j\beta) & 1 \\ f(k\alpha) & f(k\beta) & 1 \end{vmatrix} = 0; \quad (9.3')$$

$$\begin{vmatrix} f(i\alpha) & f(i\beta) & f(i\alpha)f(i\beta) & 1 \\ f(j\alpha) & f(j\beta) & f(j\alpha)f(j\beta) & 1 \\ f(k\alpha) & f(k\beta) & f(k\alpha)f(k\beta) & 1 \\ f(l\alpha) & f(l\beta) & f(l\alpha)f(l\beta) & 1 \end{vmatrix} = 0; \quad (9.4')$$

$$\begin{vmatrix} f(i\alpha) & f(i\beta) & \dots & f(i\tau) \\ f(j\alpha) & f(j\beta) & \dots & f(j\tau) \\ \dots & \dots & \dots & \dots \\ f(v\alpha) & f(v\beta) & \dots & f(v\tau) \end{vmatrix} = 0, \quad (9.5')$$

$$\begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & f(i\alpha) & f(i\beta) & \dots & f(i\tau) \\ 1 & f(j\alpha) & f(j\beta) & \dots & f(j\tau) \\ \dots & \dots & \dots & \dots & \dots \\ 1 & f(v\alpha) & f(v\beta) & \dots & f(v\tau) \end{vmatrix} = 0; \quad (9.6')$$

$$\begin{vmatrix} f(i\alpha) & f(i\beta) & \dots & f(i\tau) & 1 \\ f(j\alpha) & f(j\beta) & \dots & f(j\tau) & 1 \\ f(k\alpha) & f(k\beta) & \dots & f(k\tau) & 1 \\ \dots & \dots & \dots & \dots & \dots \\ f(v\alpha) & f(v\beta) & \dots & f(v\tau) & 1 \end{vmatrix} = 0. \quad (9.7')$$

By a motion in a geometry of two sets we understand such a pair of smooth locally invertible transformations of the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ :

$$x' = \lambda(x), \quad \xi' = \sigma(\xi), \quad (9.8)$$

under which the function (9.1) is preserved:

$$f(\lambda(x), \sigma(\xi)) = f(x, \xi). \quad (9.9)$$

If the metric function  $f$  is defined in its explicit coordinate representation (9.1), then the equality (9.9) is the functional equation with respect to the two transformations (9.8), the solution of which gives the group of motions and helps establish the number of its continuous parameters. Further in this paragraph we also give the full local groups of local motions for each of the six metric functions (9.2)–(9.7) that can be found (see [24], §2) as general solutions of the respective equations (9.9), no restrictions other than smoothness and local invertibility being imposed on the functions  $\lambda(x)$  and  $\sigma(\xi)$  of the transformations (9.8) (such as necessity of being linear etc.).

**Theorem 1.** *The group of motions (9.8) of the phenomenologically symmetric geometry of two sets (physical structure) of rank  $(n + 1, m + 1)$  defined by one of the metric functions (9.2) – (9.7) is represented by the transformations of the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  as follows:*

*for the metric function (9.2):*

$$x' = x + a, \quad \xi' = \xi - a; \quad (9.10)$$

*for the metric function (9.3):*

$$x' = ax + b, \quad \xi' = \xi/a, \quad \eta' = \eta - b\xi/a, \quad (9.11)$$

where  $a \neq 0$ ;

*for the metric function (9.4):*

$$\left. \begin{aligned} x' &= (ax + b)/(cx + d), \quad \xi' = (d\xi - c\eta)/(d - c\vartheta), \\ \eta' &= (a\eta - b\xi)/(d - c\vartheta), \quad \vartheta' = (a\vartheta - b)/(d - c\vartheta), \end{aligned} \right\} \quad (9.12)$$

where  $ad - bc = \pm 1$ ;

for the metric function (9.5):

$$\left. \begin{aligned} x'^\mu &= a^{\mu 1}x^1 + \dots + a^{\mu m}x^m, \\ \xi'^\mu &= \tilde{a}^{1\mu}\xi^1 + \dots + \tilde{a}^{m\mu}\xi^m, \end{aligned} \right\} \quad (9.13)$$

where  $\mu = 1, \dots, m$  and  $a$  is a quadratic nondegenerate matrix of degree  $m$ , and  $\tilde{a}$  is its reciprocal matrix;

for the metric function (9.6):

$$\left. \begin{aligned} x'^\nu &= a^{\nu 1}x^1 + \dots + a^{\nu, m-1}x^{m-1} + b^\nu, \\ x'^m &= x^m + c^1x^1 + \dots + c^{m-1}x^{m-1} + b^m, \\ \xi'^\nu &= \tilde{a}^{1\nu}(\xi^1 - c^1) + \dots + \tilde{a}^{m-1, \nu}(\xi^{m-1} - c^{m-1}), \\ \xi'^m &= \xi^m - (b^1\tilde{a}^{11} + \dots + b^{m-1}\tilde{a}^{1, m-1})(\xi^1 - c^1) - \dots - \\ &\quad - (b^1\tilde{a}^{m-1, 1} + \dots + b^{m-1}\tilde{a}^{m-1, m-1})(\xi^{m-1} - c^{m-1}) - b^m, \end{aligned} \right\} \quad (9.14)$$

where  $\nu = 1, \dots, m-1$  and  $a$  is a quadratic nondegenerate matrix of degree  $m-1$ , and  $\tilde{a}$  is its reciprocal matrix;

for the metric function (9.7):

$$\left. \begin{aligned} x'^\mu &= a^{\mu 1}x^1 + \dots + a^{\mu m}x^m + b^\mu, \\ \xi'^\mu &= \tilde{a}^{1\mu}\xi^1 + \dots + \tilde{a}^{m\mu}\xi^m, \\ \xi'^{m+1} &= \xi^{m+1} - (b^1\tilde{a}^{11} + \dots + b^m\tilde{a}^{1m})\xi^1 - \\ &\quad - \dots - (b^1\tilde{a}^{m1} + \dots + b^m\tilde{a}^{mm})\xi^m, \end{aligned} \right\} \quad (9.15)$$

where  $\mu = 1, \dots, m$  and  $a$  is a quadratic nondegenerate matrix of degree  $m$ , and  $\tilde{a}$  is its reciprocal matrix.

All the groups of motions represented in Theorem 1 depend on the finite number of continuous parameters which number, according to Theorem 2 of §8 is equal to  $mn$ , i.e. the direct product of the dimensionalities  $m$  and  $n$  of the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ . For the sake of comparison, we shall note that in the  $n$ -dimensional phenomenologically symmetric geometry of rank  $n+2$  on one set  $\mathfrak{M}$  that number is equal to  $n(n+1)/2$ . We shall also note that not for every metric function (9.1) the equation (9.9) has a nontrivial solution, i.e. the full group of motions may only consist of identical transformations of the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ . For example, it is easy to establish that for the metric function  $f(x, \xi) = x\xi + \xi^3$  the equation (9.9) has only a trivial solution:  $\lambda(x) = x, \sigma(\xi) = \xi$ , and so the full group of motions of the respective



geometry of two sets contains only identical transformations  $x' = x$  and  $\xi' = \xi$  of the unimetric manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ . Under Theorem 3 of §8, a geometry of two sets defined on the unimetric manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  by that metric function and endowed with such trivial group symmetry is not a physical structure of rank (2,2).

Let us scrutinize in greater detail the phenomenologically symmetric geometry of two sets (physical structure) of rank (3,3), which exists in two variants defined on two-dimensional manifolds by the metric functions

$$f = x\xi + y\eta, \quad (9.16)$$

$$f = x\xi + y + \eta, \quad (9.17)$$

whose coordinate representations are obtained from the expressions (9.5) and (9.6) for the case of  $m = n = 2$  by way of introducing non-index designations of the coordinates:  $x = x^1$ ,  $y = x^2$ ,  $\xi = \xi^1$ ,  $\eta = \xi^2$ .

It is easy to make sure that their phenomenological symmetry is expressed by the equations

$$\begin{vmatrix} f(i\alpha) & f(i\beta) & f(i\gamma) \\ f(j\alpha) & f(j\beta) & f(j\gamma) \\ f(k\alpha) & f(k\beta) & f(k\gamma) \end{vmatrix} = 0, \quad (9.16')$$

$$\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & f(i\alpha) & f(i\beta) & f(i\gamma) \\ 1 & f(j\alpha) & f(j\beta) & f(j\gamma) \\ 1 & f(k\alpha) & f(k\beta) & f(k\gamma) \end{vmatrix} = 0, \quad (9.17')$$

respectively.

We shall establish, in the first place, that these two physical structures are not equivalent.

**Theorem 2.** *Under no changes of coordinates and no scaling transformations may the metric functions (9.16) and (9.17) be transformed one into the other.*

We shall prove Theorem 2 using the method of proof by contradiction supposing that with some smooth invertible changes of coordinates  $\lambda(x, y) \rightarrow$

$x$ ,  $\sigma(x, y) \rightarrow y$  and  $\rho(\xi, \eta) \rightarrow \xi$ ,  $\tau(\xi, \eta) \rightarrow \eta$  in the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ , and a scaling transformation  $\chi(f) \rightarrow f$  one of the metric functions (9.16), (9.17) is transformed into the other, for example:

$$\lambda(x, y)\rho(\xi, \eta) + \sigma(x, y)\tau(\xi, \eta) = \chi(x\xi + y + \eta), \quad (9.18)$$

where  $\partial(\lambda, \sigma)/\partial(x, y) \neq 0$ ,  $\partial(\rho, \tau)/\partial(\xi, \eta) \neq 0$  and  $\chi' \neq 0$ . Theorem 2 will be true if the functional equation (9.18) has no solution.

We shall differentiate the equation (9.18) with respect to the variables  $\xi$ , and  $\eta$  and divide one result of the differentiation by the other:  $\lambda\rho_\xi + \sigma\tau_\xi = (\lambda\rho_\eta + \sigma\tau_\eta)x$ , wherefrom, fixing the variables  $\xi$ , and  $\eta$ , we get the relation:

$$\sigma(x, y) = A(x)\lambda(x, y), \quad (9.19)$$

where  $A(x) = (ax + b)/(cx + d)$  is a homographic function with the derivative unequal to zero:  $A'(x) = (ad - bc)/(cx + d)^2 \neq 0$ , as the functions  $\lambda$  and  $\sigma$  are independent.

Quite similarly, differentiating the equation (9.18) with respect to the variables  $x$ , and  $y$ , we get the second relation:

$$\tau(\xi, \eta) = B(\xi)\rho(\xi, \eta), \quad (9.20)$$

where  $B(\xi) = (k\xi + l)/(m\xi + n)$  is a homographic function with the derivative unequal to zero:  $B'(\xi) = (kn - lm)/(m\xi + n)^2 \neq 0$ , as the functions  $\rho$  and  $\tau$  are independent.

We shall substitute the two relations, (9.19) and (9.20), obtained into the initial functional equation (9.18):

$$\lambda(x, y)\rho(\xi, \eta)(1 + A(x)B(\xi)) = \chi(x\xi + y + \eta) \quad (9.21)$$

and differentiate it with respect to the variables  $y$  and  $\eta$ , whereafter eliminate the variable  $\chi'$ . Dividing, further, the variables, we get the differential equation  $\lambda_y/\lambda = \rho_\eta/\rho = h \neq 0$ , wherefrom, after integrating:

$$\lambda(x, y) = C(x) \exp hy, \quad \rho(\xi, \eta) = D(\xi) \exp h\eta, \quad (9.22)$$

where, obviously,  $C(x) \neq 0$ ,  $D(\xi) \neq 0$ .

We shall rewrite the equation (9.21) with the functions (9.22):

$$(1 + A(x)B(\xi))C(x)D(\xi) \exp h(y + \eta) = \chi(x\xi + y + \eta). \quad (9.23)$$

Setting  $x = 0$ ,  $\xi = 0$  and introducing the variable  $z = y + \eta$ , we obtain from the equation (9.23) the expression  $\chi(z) = E \exp hz$ , where  $E \neq 0$ , with which it becomes much simpler:

$$(1 + A(x)B(\xi))C(x)D(\xi) = E \exp hx\xi$$

We find the logarithm of that equation:

$$\ln(1 + A(x)B(\xi)) + \ln C(x) + \ln D(\xi) = \ln E + hx\xi,$$

next differentiating it with respect to the variables  $x$  and  $\xi$  performing next reduction to a common denominator:

$$A'(x)B'(\xi) = h(1 + A(x)B(\xi))^2.$$

With respect to the latter result, the same actions are repeated:

$$A'(x)B'(\xi) = 0,$$

which is, obviously, in contradiction with the non-vanishing into zero of the derivatives of the functions  $A(x)$  and  $B(\xi)$  that are part of the relations (9.19) and (9.20) that we established above. The contradiction we arrive at means that the initial functional equation (9.18) has no solution, and so the metric functions (9.16) and (9.17) are nonequivalent. The proof of Theorem 2 is complete.

Let us now find the group symmetry of the physical structure of rank (3,3) whose degree under Theorem 2 of §8 must be equal to four.

**Theorem 3.** *The group of motions of the phenomenologically symmetric geometry of two sets (physical structure) of rank (3,3) defined on two-dimensional manifolds by the metric function (9.16):  $f = x\xi + y\eta$ , is represented by the equations as follows:*

$$\left. \begin{aligned} x' &= ax + by, & y' &= cx + dy, \\ \xi' &= (d\xi - c\eta)/\Delta, & \eta' &= (-b\xi + a\eta)/\Delta, \end{aligned} \right\} \quad (9.24)$$

where  $\Delta = ad - bc \neq 0$ .

Motion in such geometry may be written using equations as follows:

$$\left. \begin{aligned} x' &= \lambda(x, y), & y' &= \sigma(x, y), \\ \xi' &= \rho(\xi, \eta), & \eta' &= \tau(\xi, \eta), \end{aligned} \right\} \quad (9.25)$$

where  $\partial(\lambda, \sigma)/\partial(x, y) \neq 0$ ,  $\partial(\rho, \tau)/\partial(\xi, \eta) \neq 0$ , as the respective transformations of the two-dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  in the motions must be invertible. Since the motion (9.25) preserves the metric function (9.16), we have for it the functional equation

$$\lambda(x, y)\rho(\xi, \eta) + \sigma(x, y)\tau(\xi, \eta) = x\xi + y\eta, \quad (9.26)$$

that is satisfied identically with respect to all the four coordinates  $x, y$  and  $\xi, \eta$ .

We shall differentiate the equation (9.26) with respect to the variables  $\xi, \eta$ :

$$\lambda\rho_\xi + \sigma\tau_\xi = x, \quad \lambda\rho_\eta + \sigma\tau_\eta = y$$

and solve the equalities obtained with respect to the functions  $\lambda, \sigma$ :

$$\lambda(x, y) = \frac{x\tau_\eta - y\tau_\xi}{\rho_\xi\tau_\eta - \rho_\eta\tau_\xi}, \quad \sigma(x, y) = \frac{-x\rho_\eta + y\rho_\xi}{\rho_\xi\tau_\eta - \rho_\eta\tau_\xi}.$$

Differentiating the expressions obtained for the functions  $\lambda$  and  $\sigma$  with respect to the variables  $x, y$ , we make sure that the coefficients with them are constants. Introducing proper designations for them, we get the first pair of equations (9.24), which define the transformation of the two-dimensional manifold  $\mathfrak{M}$  in the motion (9.25), their invertibility obviously having, as the corollary of it, the condition of  $\Delta \neq 0$ . The other pair of equations (9.24), which defines the transformation of the two-dimensional manifold  $\mathfrak{N}$  in the motion (9.25), is easily come by from the functional equation (9.26) by way of substituting the former pair of equations into it.

The set of motions (9.24) depends on the four continuous parameters  $a, b, c, d$ , with the condition imposed upon them of  $\Delta = ad - bc \neq 0$ . It is easy to make sure that that set, by its composition of motions, is a group. To do it, let us write, for example, the transformations of the former manifold,  $\mathfrak{M}$ , in the set of motions (9.24) in the matrix form:

$$\begin{vmatrix} x' \\ y' \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix}.$$

That is, every such transformation is univocally assigned a quadratic nondegenerate matrix of second order, and the composition of the two transformations their matrix multiplication according to the "row by column" rule. It is well-known that the set of all nondegenerate quadratic matrices in the operation

of their ordinary multiplication is a group, so the set of transformations of the manifold  $\mathfrak{M}$  in the set of motions (9.24) is a group too. The transformations of the manifold  $\mathfrak{N}$  in these motions are assigned the reciprocal transposes of matrices, whose set is also a group, which is isomorphic with respect to the group of the initial matrices. Therefore, the whole set of motions (9.24), as a conglomerate of isomorphic groups of transformations of different manifolds is a group that gives the group symmetry of the physical structure of rank (3,3) defined by the metric function (9.16). The degree of that symmetry is equal to 4, because the group of motions (9.24) depends on four continuous and independent parameters. Theorem 3 is proved.

**Theorem 4.** *The group of motions of a phenomenologically symmetric geometry of two sets (physical structure) of rank (3,3) defined on two-dimensional manifolds by the metric function (9.17):  $f = x\xi + y + \eta$ , is represented by the following equations:*

$$\left. \begin{aligned} x' &= ax + b, & y' &= y + cx + d, \\ \xi' &= (\xi - c)/a, & \eta' &= \eta - b\xi/a - (ad - bc)/a, \end{aligned} \right\} \quad (9.27)$$

where  $a \neq 0$ .

We shall write the functional equation on the set of motions (9.25) for the metric function (9.17):

$$\lambda(x, y)\rho(\xi, \eta) + \sigma(x, y) + \tau(\xi, \eta) = x\xi + y + \eta \quad (9.28)$$

and differentiate it with respect to the variables  $\xi$  and  $\eta$ :

$$\lambda\rho_\xi + \tau_\xi = x, \quad \lambda\rho_\eta + \tau_\eta = 1,$$

which yields:  $\lambda(x, y) = (x\tau_\eta - \tau_\xi)/(\rho_\xi\tau_\eta - \rho_\eta\tau_\xi)$ . We shall fix in the right-hand member the variables  $\xi, \eta$  and introduce suitable designation for the constant coefficients:  $\lambda(x, y) = ax + b$ , where  $a \neq 0$ , as  $\lambda(x, y) \neq \text{const.}$  By substituting that expression for the function  $\lambda$  in the initial functional equation (9.28) and fixing again the variables  $\xi, \eta$  we get an expression for the other function  $\sigma(x, y) = y + cx + d$ . Thus we have obtained transformations for the manifold  $\mathfrak{M}$  in the set of motions (9.27). The transformations for the second manifold,  $\mathfrak{N}$ , are found from the functional equation (9.28) by

way of substituting the expressions for the functions  $\lambda$  and  $\sigma$  that we have just found. Which gives us the full set of motions (9.27).

In order to make certain that the set of motions (9.27) is a group, let us write, for example, the transformation of a set  $\mathfrak{M}$  from it in the matrix form as follows:

$$\begin{vmatrix} x' \\ y' \\ 1 \end{vmatrix} = \begin{vmatrix} a & 0 & b \\ c & 1 & d \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ 1 \end{vmatrix},$$

which demonstrates that every such transformation is assigned a nondegenerate matrix of third order, whose structure is obviously preserved under usual matrix "row by column" multiplication, the composition of the two transformations being assigned the product of the matrices. The set of nondegenerate matrices of such structure is a group under the operation of their multiplication, and so the set of transformations of the manifold  $\mathfrak{M}$  in the motions (9.27) is also a group. It is not difficult to see that the transformations of the manifold  $\mathfrak{M}$  in these motions are assigned the transposed reciprocal matrices whose set is also a group, isomorphic to the group of the initial matrices. Thus, the whole set of motions (9.27) is a group that determines the symmetry of the geometry of two sets of rank (3,3) that is defined on two-dimensional manifolds by the metric function (9.17). The degree of the group symmetry equals 4, as the group of motions (9.27) depends on the four continuous and independent parameters. Theorem 4 is proved.

**Theorem 5.** *The two-point invariant of the group of transformations (9.24) coincides with the metric function (9.16) with an accuracy up to a scaling transformation.*

The parameters of the identity transformation in the group (9.24) are  $a = 1$ ,  $b = 0$ ,  $c = 0$ ,  $d = 1$ . We shall introduce the parameters of the indefinitely small (infinitesimal) transformation  $\alpha, \beta, \gamma, \delta$ , setting  $a = 1 + \alpha$ ,  $b = \beta$ ,  $c = \gamma$ ,  $d = 1 + \delta$ . Then, with an accuracy up to values of

the first order of smallness the transformations (9.24) will be as follows:

$$\left. \begin{aligned} x' &= x + \alpha x + \beta y, & y' &= y + \gamma x + \delta y, \\ \xi' &= \xi - \alpha\xi - \gamma\eta, & \eta' &= \eta - \beta\xi - \delta\eta. \end{aligned} \right\} \quad (9.29)$$

The infinitely small transformations (9.29) may be assigned two systems of four linear differential operators:

$$\left. \begin{aligned} X_1 &= x\partial_x, & X_2 &= y\partial_x, & X_3 &= x\partial_y, & X_4 &= y\partial_y, \\ \Xi_1 &= -\xi\partial_\xi, & \Xi_2 &= -\xi\partial_\eta, & \Xi_3 &= -\eta\partial_\xi, & \Xi_4 &= -\eta\partial_\eta, \end{aligned} \right\} \quad (9.30)$$

where, for example,  $\partial_x = \partial/\partial x$  which comprise natural coordinate bases of the two isomorphic with an accuracy up to the coincidence of the structural constants four-dimensional Lie algebras of the transformations of the two-dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ .

With the known transformations (9.24), the two-point invariant  $f = f(x, y, \xi, \eta)$  is the solution of the functional equation

$$f(x', y', \xi', \eta') = f(x, y, \xi, \eta). \quad (9.31)$$

If we substitute in the functional equation (9.31) the infinitely small transformations (9.29), differentiate it with respect to each of the four parameters  $\alpha, \beta, \gamma, \delta$  and assign zero values to them, then with respect to the two-point invariant a system of four differential equations

$$X_\omega f + \Xi_\omega f = 0, \quad (9.32)$$

appears where  $\omega = 1, 2, 3, 4$ , with the operators (9.30).

Since the differential equations (9.32) are ones linear homogeneous in the partial derivatives of the second order, they may be solved by the method of characteristics. For the first and the fourth equations of the system, the respective equations of characteristics are  $dx/x = -d\xi/\xi$ ,  $dy/y = -d\eta/\eta$  and have the integrals  $x\xi = \text{const}$ ,  $y\eta = \text{const}$ . We shall substitute the general solution  $f = \theta(x\xi, y\eta)$  of the first and the fourth equations of the system (9.32), where  $\theta(u, v)$  is an arbitrary function of two variables, into the second and third equations:  $\theta_u - \theta_v = 0$ . This equation is also solved by the method of characteristics, and its general solution is the expression  $\theta(u, v) = \chi(u+v)$ , where  $\chi$  is an arbitrary function of only one variable with the derivative  $\chi'$  which is unequal to zero. Thus, the two-point invariant, as

the solution of the system of differential equations (9.32) with the operators (9.30) is

$$f = \chi(x\xi + y\eta), \quad (9.33)$$

which is transformed by the scaling transformation  $\chi^{-1}(f) \rightarrow f$  with the inverse  $\chi^{-1}$  into the metric function (9.16). Theorem 5 is proved.

**Theorem 6.** *The two-point invariant of the group of transformations (9.27) coincides with the metric function (9.17) with an accuracy up to a scaling transformation.*

The proof of Theorem 6 mainly repeats the proof of Theorem 5, differing from it, grosso modo, in some small detail. The identity transformation in the group (9.27) will be the transformation with the parameters  $a = 1$ ,  $b = 0$ ,  $c = 0$ ,  $d = 0$ . Setting  $a = 1 + \alpha$ ,  $b = \beta$ ,  $c = \gamma$ ,  $d = \delta$ , with an accuracy up to the terms of the first order of smallness from the equation (9.27) we get equations for infinitely small (infinitesimal) transformations:

$$\left. \begin{aligned} x' &= x + \alpha x + \beta, & y' &= y + \gamma x + \delta, \\ \xi' &= \xi - \alpha\xi - \gamma, & \eta' &= \eta - \beta\xi - \delta, \end{aligned} \right\} \quad (9.34)$$

which correspond two systems of four linear differential operators:

$$\left. \begin{aligned} X_1 &= x\partial_x, & X_2 &= \partial_x, & X_3 &= x\partial_y, & X_4 &= \partial_y, \\ \Xi_1 &= -\xi\partial_\xi, & \Xi_2 &= -\xi\partial_\eta, & \Xi_3 &= -\partial_\xi, & \Xi_4 &= -\partial_\eta, \end{aligned} \right\} \quad (9.35)$$

which comprise the natural coordinate bases of two isomorphic with an accuracy up to the coincidence of the structural constants four-dimensional Lie algebras of the transformations (9.27) of the two-dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ .

We shall substitute into the functional equation (9.31) for the two-point invariant the infinitesimal transformations (9.34), differentiate it with respect to each of the four parameters  $\alpha, \beta, \gamma, \delta$  and in the results of the differentiation assign them zero values corresponding the identity transformations. That yields for the two-point invariant  $f = f(x, y, \xi, \eta)$  the system of the four differential equations (9.32) with the operators (9.35). As is in the previous case, the equations are solved with the method of characteristics. For the first and the fourth equations of the system (9.32) the respective equations



of characteristics,  $dx/x = -d\xi/\xi$ ,  $dy = -d\eta$ , are easily integrable:  $x\xi = \text{const}$ ,  $y + \eta = \text{const}$ , and their general solution will be written as follows:  $f = \theta(x\xi, y + \eta)$ , where  $\theta(u, v)$  is an arbitrary function of two variables. The substitution of that expression into the second and third equations of the system yields the differential equation  $\theta_u - \theta_v = 0$  whose solution  $\theta(u, v) = \chi(u + v)$  is written via the arbitrary function  $\chi$  of only one variable,  $\chi'$  being unequal to zero. As result, for the two-point invariant  $f$  we get the expression

$$f = \chi(x\xi + y + \eta), \quad (9.36)$$

that is transformed into the metric function (9.17) through the scaling transformation  $\chi^{-1}(f) \rightarrow f$  with the inverse function  $\chi^{-1}$ . Theorem 6 is proved.

We shall note that the two four-dimensional Lie algebras (9.30) in the respective bases have the same structural constants. The obvious transition to the other basis coupled with a trivial change of coordinates transforms one basis into the other, which means the weak equivalence of the algebras. However, no change of coordinates *alone* transforms the bases one into the other. That circumstance implies that the two corresponding groups of Lie transformations (9.24), as different actions in the two-dimensional manifold of one and the same four-dimensional Lie group, are similar but not equivalent. That is, some automorphism in the group accompanied by the change of coordinates *will* transform one group of transformations into the other (similarity, or weak equivalence), but no change of coordinates *without* automorphism can do the same (nonequivalence in the strong sense). The same is to be said as concerns the two four-dimensional Lie algebras (9.35) that correspond the Lie transformations (9.27).

## §10. Dimetric and trimetric physical structures

The full classification of the dimetric physical structures (phenomenologically symmetric geometries of two sets) has been only built for the rank

$(n + 1, 2)$ . A brief definition of theirs is derived from Definition 1 of the polymetric physical structures of rank  $(n + 1, m + 1)$ , that was given at the beginning of §8, if we set  $s = 2$  and  $m = 1$  in it.

Suppose there are two sets,  $\mathfrak{M}$  and  $\mathfrak{N}$ , that are a two-dimensional and a  $2n$ -dimensional manifolds respectively, where  $n$  is a natural number. We shall designate the local coordinates in the manifolds as  $x = (x^1, x^2)$  and  $\xi = (\xi^1, \dots, \xi^{2n})$ . Suppose there is also a function  $f$  with the domain  $\mathfrak{S}_f$  open and dense in  $\mathfrak{M} \times \mathfrak{N}$  that assigns to every pair of it two real numbers, i.e.  $f : \mathfrak{S}_f \rightarrow R^2$ . We shall call the two-point two-component function  $f = (f^1, f^2)$  a *metric* one. It is supposed that its local coordinate representation is defined by a sufficiently smooth nondegenerate function

$$f = f(x, \xi) = f(x^1, x^2, \xi^1, \dots, \xi^{2n}), \quad (10.1)$$

the expression for which is obtained from the expression (8.2) with  $s = 2$  and  $m = 1$ . The nondegeneracy of the metric function (10.1) is understood in the sense of Axiom III of §8 and, generally speaking, and in contrast to the case of  $s = 1$ , i.e. that of unimetric physical structures, means somewhat more than its mere essential dependence on the coordinates  $x = (x^1, x^2)$  and  $\xi = (\xi^1, \dots, \xi^{2n})$ . And that is the necessary nonzero quality of the Jacobians  $\partial f(i\alpha)/\partial x_i$  and  $\partial(f(i_1\alpha), \dots, f(i_n\alpha))/\partial \xi_\alpha$  for the dense sets of pairs  $\langle i\alpha \rangle \in \mathfrak{M} \times \mathfrak{N}$  and corteges  $\langle i_1 \dots i_n, \alpha \rangle \in \mathfrak{M}^n \times \mathfrak{N}$  of length  $n + 1$ . Further, we build the function  $F$  with the natural in  $\mathfrak{M}^{n+1} \times \mathfrak{N}^2$  domain  $\mathfrak{S}_F$  by assigning to every cortege  $n + 3$  from  $\mathfrak{S}_F$  all the  $4(n + 1)$  distances possible in the metric function  $f = (f^1, f^2)$ . We shall say that the two-component function  $f$  with the local coordinate representation (10.1) gives on a two-dimensional manifold  $\mathfrak{M}$  and a  $2n$ -dimensional manifold  $\mathfrak{N}$  a *dimetric physical structure* (phenomenologically symmetric geometry of two sets) of rank  $(n + 1, 2)$ , if locally the set of values  $F(\mathfrak{S}_F)$  in  $R^{4(n+1)}$  belongs to the set of zeros of some sufficiently smooth two-component function  $\Phi = (\Phi_1, \Phi_2)$  of  $4(n + 1)$  variables with the independent components  $\Phi_1$  and  $\Phi_2$ , i.e. an equation

$$\Phi(f(i\alpha), f(i\beta), f(j\alpha), f(j\beta), \dots, f(v\alpha), f(v\beta)) = 0 \quad (10.2)$$

takes place for all the corteges  $\langle ijk \dots v, \alpha\beta \rangle$  from a set dense and open

in  $\mathfrak{S}_F \subseteq \mathfrak{M}^{n+1} \times \mathfrak{N}^2$ . Thus, the set  $F(\mathfrak{S}_F)$  locally belongs to some regular surface in  $R^{4(n+1)}$ , of codimension 2, not necessarily coinciding with it.

We shall note that not every two-component function  $f = (f^1, f^2)$  may give a dimetric physical structure, so the principle task for the theory is their complete classification that is to be carried out with an accuracy, as usual, up to a scaling transformation, two-dimensional in this case, and the possibility of choosing in the manifold  $\mathfrak{M}$  and  $\mathfrak{N}$  of any allowable systems of local coordinates.

**Theorem 1.** *Dimetric physical structures (phenomenologically symmetric geometries of two sets) of rank  $(n+1, 2)$  exist only for  $n = 1, 2, 3, 4$ , that is rank  $(2, 2)$ ,  $(3, 2)$ ,  $(4, 2)$ ,  $(5, 2)$ , and do not exist for  $n \geq 5$ , i.e. for rank  $(6, 2)$ ,  $(7, 2)$  etc. With an accuracy up to a scaling transformation the two-component metric function  $f = (f^1, f^2)$  that defines on a 2-dimensional and a  $2n$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a dimetric physical structure of rank  $(n+1, 2)$  in systems of local coordinates  $x = (x^1, x^2) = (x, y)$  and  $\xi = (\xi^1, \xi^2, \xi^3, \xi^4, \dots) = (\xi, \eta, \mu, \nu, \dots)$  suitably chosen in them is defined with the following canonical expressions:*

for  $n = 1$ , that is for rank  $(2, 2)$ :

$$f^1 = x + \xi, \quad f^2 = y + \eta, \quad (10.3)$$

$$f^1 = (x + \xi)y, \quad f^2 = (x + \xi)\eta; \quad (10.4)$$

for  $n = 2$ , i.e. for rank  $(3, 2)$ :

$$f^1 = x\xi + \varepsilon y\eta + \mu, \quad f^2 = x\eta + y\xi + \nu, \quad \varepsilon = 0, \pm 1, \quad (10.5)$$

$$f^1 = x\xi + \mu, \quad f^2 = x\eta + y\xi^c + \nu, \quad c \neq 1, \quad (10.6)$$

$$f^1 = x\xi + \mu, \quad f^2 = x\eta + y\xi^2 + x^2\xi^2 \ln \xi + \nu, \quad (10.7)$$

$$f^1 = x\xi + y\mu, \quad f^2 = x\eta + y\nu; \quad (10.8)$$

for  $n = 3$ , i.e. for rank  $(4, 2)$ :

$$\left. \begin{aligned} f^1 &= \frac{(x\xi + \varepsilon y\eta + \mu)(x + \rho) - \varepsilon(x\eta + y\xi + \nu)(y + \tau)}{(x + \rho)^2 - \varepsilon(y + \tau)^2}, \\ f^2 &= \frac{(x\xi + \varepsilon y\eta + \mu)(y + \tau) - (x\eta + y\xi + \nu)(x + \rho)}{(x + \rho)^2 - \varepsilon(y + \tau)^2}, \end{aligned} \right\} \quad (10.9)$$

where  $\varepsilon = 0, \pm 1$ ,

$$f^1 = \frac{x\xi + \mu}{x + \rho}, \quad f^2 = \frac{x\eta + y\nu + \tau}{x + \rho}, \quad (10.10)$$

$$f^1 = x\xi + y\mu + \rho, \quad f^2 = x\eta + y\nu + \tau; \quad (10.11)$$

for  $n = 4$ , i.e. for rank  $(5, 2)$ :

$$f^1 = \frac{x\xi + y\mu + \rho}{x\varphi + y + \omega}, \quad f^2 = \frac{x\eta + y\nu + \tau}{x\varphi + y + \omega}. \quad (10.12)$$

The proof of Theorem 1 may be found in §7 of the author's monograph [24] and in his note [26].

Let us now take the equation (10.2), that expresses the phenomenological symmetry of the dimetric physical structure of rank  $(n+1, 2)$ . We shall write it explicitly for each of the metric functions (10.3) to (10.12) respectively:

for the metric function (10.3):

$$\left. \begin{aligned} f^1(i\alpha) - f^1(i\beta) - f^1(j\alpha) + f^1(j\beta) &= 0, \\ f^2(i\alpha) - f^2(i\beta) - f^2(j\alpha) + f^2(j\beta) &= 0; \end{aligned} \right\}$$

for the metric function (10.4):

$$\left. \begin{aligned} \begin{vmatrix} f^1(i\alpha) - f^1(i\beta) & f^1(i\alpha)f^2(j\alpha) \\ f^1(j\alpha) - f^1(j\beta) & f^1(j\alpha)f^2(i\alpha) \end{vmatrix} &= 0, \\ \begin{vmatrix} f^2(i\alpha) - f^2(j\alpha) & f^2(i\alpha)f^1(i\beta) \\ f^2(i\beta) - f^2(j\beta) & f^2(i\beta)f^1(i\alpha) \end{vmatrix} &= 0; \end{aligned} \right\}$$

for the metric function (10.5):

$$\left. \begin{aligned} \begin{vmatrix} f^1(i\alpha) & f^1(i\beta) & 1 \\ f^1(j\alpha) & f^1(j\beta) & 1 \\ f^1(k\alpha) & f^1(k\beta) & 1 \end{vmatrix} + \varepsilon \begin{vmatrix} f^2(i\alpha) & f^2(i\beta) & 1 \\ f^2(j\alpha) & f^2(j\beta) & 1 \\ f^2(k\alpha) & f^2(k\beta) & 1 \end{vmatrix} &= 0, \\ \begin{vmatrix} f^1(i\alpha) & f^2(i\beta) & 1 \\ f^1(j\alpha) & f^2(j\beta) & 1 \\ f^1(k\alpha) & f^2(k\beta) & 1 \end{vmatrix} + \begin{vmatrix} f^2(i\alpha) & f^1(i\beta) & 1 \\ f^2(j\alpha) & f^1(j\beta) & 1 \\ f^2(k\alpha) & f^1(k\beta) & 1 \end{vmatrix} &= 0; \end{aligned} \right\}$$

for the metric function (10.6):

$$\hat{\mathbf{R}}(\alpha\beta) \frac{\begin{vmatrix} f^1(i\alpha) & f^2(i\alpha) & 1 \\ f^1(j\alpha) & f^2(j\alpha) & 1 \\ f^1(k\alpha) & f^2(k\alpha) & 1 \end{vmatrix}}{(f^1(i\alpha) - f^1(j\alpha))^{c+1}} = 0, \quad \begin{vmatrix} f^1(i\alpha) & f^1(i\beta) & 1 \\ f^1(j\alpha) & f^1(j\beta) & 1 \\ f^1(k\alpha) & f^1(k\beta) & 1 \end{vmatrix} = 0,$$

where  $\hat{\mathbf{R}}(\alpha\beta)$  is the operator of alternation (antisymmetrization) with respect to the elements  $\alpha, \beta$ , i.e.  $\hat{\mathbf{R}}(\alpha\beta)\varphi(\alpha\beta) = \varphi(\alpha\beta) - \varphi(\beta\alpha)$ ;

for the metric function (10.7):

$$\left. \begin{aligned} & \begin{vmatrix} f^1(i\alpha) & f^1(i\beta) & 1 \\ f^1(j\alpha) & f^1(j\beta) & 1 \\ f^1(k\alpha) & f^1(k\beta) & 1 \end{vmatrix} = 0, \\ & \hat{\mathbf{R}}(\alpha\beta) \frac{1}{(f^1(i\alpha) - f^1(j\alpha))^3} \left\{ \begin{vmatrix} f^1(i\alpha) & f^2(i\alpha) & 1 \\ f^1(j\alpha) & f^2(j\alpha) & 1 \\ f^1(k\alpha) & f^2(k\alpha) & 1 \end{vmatrix} - \right. \\ & \left. - \begin{vmatrix} f^1(i\alpha) & (f^1(i\alpha))^2 & 1 \\ f^1(j\alpha) & (f^1(j\alpha))^2 & 1 \\ f^1(k\alpha) & (f^1(k\alpha))^2 & 1 \end{vmatrix} \ln[f^1(i\alpha) - f^1(j\alpha)] \right\} = 0; \end{aligned} \right\}$$

for the metric function (10.8):

$$\left. \begin{aligned} & \hat{\mathbf{R}}(ij) \begin{vmatrix} f^1(i\alpha) & f^1(i\beta) \\ f^1(k\alpha) & f^1(k\beta) \end{vmatrix} \times \begin{vmatrix} f^2(j\alpha) & f^2(j\beta) \\ f^2(k\alpha) & f^2(k\beta) \end{vmatrix} = 0, \\ & \hat{\mathbf{R}}(ik) \begin{vmatrix} f^1(i\alpha) & f^1(i\beta) \\ f^1(j\alpha) & f^1(j\beta) \end{vmatrix} \times \begin{vmatrix} f^2(k\alpha) & f^2(k\beta) \\ f^2(j\alpha) & f^2(j\beta) \end{vmatrix} = 0; \end{aligned} \right\}$$

for the metric function (10.9) the equation (10.2) may be obtained by way of complexification of the equation

$$\begin{vmatrix} f(i\alpha) & f(i\beta) & f(i\alpha)f(i\beta) & 1 \\ f(j\alpha) & f(j\beta) & f(j\alpha)f(j\beta) & 1 \\ f(k\alpha) & f(k\beta) & f(k\alpha)f(k\beta) & 1 \\ f(l\alpha) & f(l\beta) & f(l\alpha)f(l\beta) & 1 \end{vmatrix} = 0,$$

by setting  $f = f^1 + ef^2$ , where  $e^2 = \varepsilon = 0, \pm 1$ , and separating the real and the imaginary part;

for the metric function (10.10):

$$\left. \begin{aligned} & \hat{R}(\alpha\beta) \frac{(f^1(j\alpha) - f^1(l\alpha))(f^1(i\alpha) - f^1(k\alpha))}{(f^1(i\alpha) - f^1(l\alpha))(f^1(j\alpha) - f^1(k\alpha))} = 0, \\ & \hat{R}(\alpha\beta) \frac{f^1(j\alpha) - f^1(l\alpha)}{f^1(i\alpha) - f^1(l\alpha)} \times \\ & \times \frac{\begin{vmatrix} f^1(i\alpha) & f^1(k\alpha) - f^1(l\alpha) \\ f^2(i\alpha) & f^2(k\alpha) - f^2(l\alpha) \end{vmatrix} + \begin{vmatrix} f^1(k\alpha) & f^1(l\alpha) \\ f^2(k\alpha) & f^2(l\alpha) \end{vmatrix}}{\begin{vmatrix} f^1(j\alpha) & f^1(k\alpha) - f^1(l\alpha) \\ f^2(j\alpha) & f^2(k\alpha) - f^2(l\alpha) \end{vmatrix} + \begin{vmatrix} f^1(k\alpha) & f^1(l\alpha) \\ f^2(k\alpha) & f^2(l\alpha) \end{vmatrix}} = 0; \end{aligned} \right\}$$

for the metric function (10.11):

$$\left. \begin{aligned} & \hat{\mathbf{R}}(ij) \begin{vmatrix} f^1(i\alpha) & f^1(i\beta) & 1 \\ f^1(k\alpha) & f^1(k\beta) & 1 \\ f^1(l\alpha) & f^1(l\beta) & 1 \end{vmatrix} \times \begin{vmatrix} f^2(j\alpha) & f^2(j\beta) & 1 \\ f^2(k\alpha) & f^2(k\beta) & 1 \\ f^2(l\alpha) & f^2(l\beta) & 1 \end{vmatrix} = 0, \\ & \hat{\mathbf{R}}(kl) \begin{vmatrix} f^1(i\alpha) & f^1(i\beta) & 1 \\ f^1(j\alpha) & f^1(j\beta) & 1 \\ f^1(k\alpha) & f^1(k\beta) & 1 \end{vmatrix} \times \begin{vmatrix} f^2(i\alpha) & f^2(i\beta) & 1 \\ f^2(j\alpha) & f^2(j\beta) & 1 \\ f^2(l\alpha) & f^2(l\beta) & 1 \end{vmatrix} = 0; \end{aligned} \right\}$$

for the metric function (10.12), the one that gives an only physical structure

of rank (5,2):

$$\left. \begin{aligned} \hat{R}(\alpha\beta) \frac{\begin{vmatrix} f^1(i\alpha) & f^2(i\alpha) & 1 \\ f^1(k\alpha) & f^2(k\alpha) & 1 \\ f^1(l\alpha) & f^2(l\alpha) & 1 \end{vmatrix}}{\begin{vmatrix} f^1(j\alpha) & f^2(j\alpha) & 1 \\ f^1(k\alpha) & f^2(k\alpha) & 1 \\ f^1(l\alpha) & f^2(l\alpha) & 1 \end{vmatrix}} \times \frac{\begin{vmatrix} f^1(j\alpha) & f^2(j\alpha) & 1 \\ f^1(k\alpha) & f^2(k\alpha) & 1 \\ f^1(m\alpha) & f^2(m\alpha) & 1 \end{vmatrix}}{\begin{vmatrix} f^1(i\alpha) & f^2(i\alpha) & 1 \\ f^1(k\alpha) & f^2(k\alpha) & 1 \\ f^1(l\alpha) & f^2(l\alpha) & 1 \end{vmatrix}} = 0, \\ \hat{R}(\alpha\beta) \frac{\begin{vmatrix} f^1(i\alpha) & f^2(i\alpha) & 1 \\ f^1(k\alpha) & f^2(k\alpha) & 1 \\ f^1(l\alpha) & f^2(l\alpha) & 1 \end{vmatrix}}{\begin{vmatrix} f^1(j\alpha) & f^2(j\alpha) & 1 \\ f^1(k\alpha) & f^2(k\alpha) & 1 \\ f^1(l\alpha) & f^2(l\alpha) & 1 \end{vmatrix}} \times \frac{\begin{vmatrix} f^1(j\alpha) & f^2(j\alpha) & 1 \\ f^1(l\alpha) & f^2(l\alpha) & 1 \\ f^1(m\alpha) & f^2(m\alpha) & 1 \end{vmatrix}}{\begin{vmatrix} f^1(i\alpha) & f^2(i\alpha) & 1 \\ f^1(l\alpha) & f^2(l\alpha) & 1 \\ f^1(m\alpha) & f^2(m\alpha) & 1 \end{vmatrix}} = 0. \end{aligned} \right\}$$

It turns out that the relation of the metric function (10.1), that gives a dimetric physical structure of rank  $(n+1, 2)$ , and the equation (10.2), that expresses its phenomenological symmetry, may be made more transparent, which is demonstrated by the following theorem proved by R.M. Muradov [32, §18]:

**Theorem 2.** *If a two-component metric function*

$$f = f(x, y, \xi, \eta, \mu, \nu, \dots)$$

*gives on a two-dimensional and a  $2n$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a dimetric physical structure (a phenomenologically symmetric geometry of two sets) of rank  $(n+1, 2)$ , then with an accuracy up to a scaling transformation and change of coordinates in the manifolds it defines in  $R^{2n}$  such a quasigroup operation with a right identity that the right inverse coincides with the parent element, and in the equation expressing the phenomenological symmetry under the operator of alternation  $\hat{R}(\alpha\beta)$  there stands an expression similar to the metric function itself:*

$$\hat{R}(\alpha\beta) f(f^1(i\alpha), f^2(i\alpha), f^1(j\alpha), f^2(j\alpha), f^1(k\alpha), f^2(k\alpha), \dots) = 0;$$

for  $n = 1$ , i.e. for rank  $(2, 2)$ :

$$f^1 = x - \xi, \quad f^2 = y - \eta, \quad (10.3')$$

$$f^1 = (x - \xi)\eta, \quad f^2 = y/\eta; \quad (10.4')$$

for  $n = 2$ , i.e. for rank  $(3, 2)$ :

$$f^1 = \frac{\begin{vmatrix} x & \xi - \mu \\ \mu & \xi - \mu \end{vmatrix} - \varepsilon \begin{vmatrix} y & \eta - \nu \\ \nu & \eta - \nu \end{vmatrix}}{(\xi - \mu)^2 - \varepsilon(\eta - \nu)^2}, \quad f^2 = \frac{\begin{vmatrix} y & x & 1 \\ \eta & \xi & 1 \\ \nu & \mu & 1 \end{vmatrix}}{(\xi - \mu)^2 - \varepsilon(\eta - \nu)^2}, \quad (10.5')$$

where  $\varepsilon = 0, \pm 1$ ;

$$f^1 = \frac{x - \mu}{\xi - \mu}, \quad f^2 = \frac{\begin{vmatrix} y & x & 1 \\ \eta & \xi & 1 \\ \nu & \mu & 1 \end{vmatrix}}{(\xi - \mu)^{c+1}}, \quad (10.6')$$

where  $c \neq 1$ ;

$$f^1 = \frac{x - \mu}{\xi - \mu}, \quad f^2 = \frac{\begin{vmatrix} y & x & 1 \\ \eta & \xi & 1 \\ \nu & \mu & 1 \end{vmatrix} - \ln(\xi - \mu) \begin{vmatrix} x^2 & x & 1 \\ \xi^2 & \xi & 1 \\ \mu^2 & \mu & 1 \end{vmatrix}}{(\xi - \mu)^3}, \quad (10.7')$$

$$f^1 = \frac{x\nu - y\mu}{\xi\nu - \eta\mu}, \quad f^2 = \frac{x\eta - y\xi}{\xi\nu - \eta\mu}; \quad (10.8')$$



for  $n = 3$ , i.e. for rank  $(4, 2)$ :

$$\left. \begin{aligned}
 f^1 &= \frac{\left| \begin{array}{c} \frac{(x-\mu)(\xi-\rho) + \varepsilon(y-\nu)(\eta-\tau)}{(x-\rho)(\eta-\nu) + (y-\tau)(\xi-\mu)} \\ \frac{\varepsilon((x-\mu)(\eta-\tau) + (y-\nu)(\xi-\rho))}{(x-\rho)(\xi-\mu) + \varepsilon(y-\tau)(\eta-\nu)} \end{array} \right|}{\left| \begin{array}{c} \frac{(x-\rho)(\xi-\mu) + \varepsilon(y-\tau)(\eta-\nu)}{(x-\rho)(\xi-\mu) + \varepsilon(y-\tau)(\eta-\nu)} \\ \frac{\varepsilon((x-\rho)(\eta-\nu) + (y-\tau)(\xi-\mu))}{(x-\rho)(\xi-\mu) + \varepsilon(y-\tau)(\eta-\nu)} \end{array} \right|}, \\
 f^2 &= \frac{\left| \begin{array}{cccc} x^2 & y & x & 1 \\ \xi^2 & \eta & \xi & 1 \\ \mu^2 & \nu & \mu & 1 \\ \rho^2 & \tau & \rho & 1 \end{array} \right| - \varepsilon \left| \begin{array}{cccc} y^2 & y & x & 1 \\ \eta^2 & \eta & \xi & 1 \\ \nu^2 & \nu & \mu & 1 \\ \tau^2 & \tau & \rho & 1 \end{array} \right|}{((x-\rho)^2 - \varepsilon(y-\tau)^2)((\xi-\mu)^2 - \varepsilon(\eta-\nu)^2)},
 \end{aligned} \right\} \quad (10.9')$$

where  $\varepsilon = 0, \pm 1$ ;

$$f^1 = \frac{(x-\mu)(\xi-\rho)}{(x-\rho)(\xi-\mu)}, \quad f^2 = \frac{(\mu-\rho)(\xi-\rho)}{(x-\rho)(\xi-\mu)} \cdot \frac{\left| \begin{array}{ccc} y & x & 1 \\ \eta & \xi & 1 \\ \nu & \mu & 1 \end{array} \right|}{\left| \begin{array}{ccc} \xi & \eta & 1 \\ \mu & \nu & 1 \\ \rho & \tau & 1 \end{array} \right|}, \quad (10.10')$$

$$f^1 = \frac{\left| \begin{array}{ccc} x & y & 1 \\ \mu & \nu & 1 \\ \rho & \tau & 1 \end{array} \right|}{\left| \begin{array}{ccc} \xi & \eta & 1 \\ \mu & \nu & 1 \\ \rho & \tau & 1 \end{array} \right|}, \quad f^2 = \frac{\left| \begin{array}{ccc} x & y & 1 \\ \xi & \eta & 1 \\ \rho & \tau & 1 \end{array} \right|}{\left| \begin{array}{ccc} \xi & \eta & 1 \\ \mu & \nu & 1 \\ \rho & \tau & 1 \end{array} \right|}; \quad (10.11')$$

for  $n = 4$ , i.e. for rank  $(5, 2)$ :

$$f^1 = \frac{\begin{vmatrix} x & y & 1 \\ \mu & \nu & 1 \\ \varphi & \omega & 1 \end{vmatrix}}{\begin{vmatrix} \xi & \eta & 1 \\ \mu & \nu & 1 \\ \varphi & \omega & 1 \end{vmatrix}} \cdot \frac{\begin{vmatrix} \xi & \eta & 1 \\ \rho & \tau & 1 \\ \varphi & \omega & 1 \end{vmatrix}}{\begin{vmatrix} x & y & 1 \\ \rho & \tau & 1 \\ \varphi & \omega & 1 \end{vmatrix}}, \quad f^2 = \frac{\begin{vmatrix} x & y & 1 \\ \xi & \eta & 1 \\ \rho & \tau & 1 \end{vmatrix}}{\begin{vmatrix} \xi & \eta & 1 \\ \mu & \nu & 1 \\ \rho & \tau & 1 \end{vmatrix}} \cdot \frac{\begin{vmatrix} \mu & \nu & 1 \\ \rho & \tau & 1 \\ \varphi & \omega & 1 \end{vmatrix}}{\begin{vmatrix} x & y & 1 \\ \rho & \tau & 1 \\ \varphi & \omega & 1 \end{vmatrix}}, \quad (10.12')$$

The complete classification of the trimetric physical structures (phenomenologically symmetric geometries of two sets) has only been built for rank  $(2, 2)$ . Such a structure, according to general Definition 1 of §8 where we must set  $s = 3$ ,  $m = 1$ ,  $n = 1$ , is defined by a three-component function  $f = (f^1, f^2, f^3)$  on three-dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ . We shall designate the local coordinates in these manifolds by  $x, y, z$  and  $\xi, \eta, \vartheta$ . Then the coordinate representation of the metric function  $f$  is written as follows:

$$f = f(x, y, z, \xi, \eta, \vartheta), \quad (10.13)$$

for any pair  $\langle i\alpha \rangle$  from the domain  $\mathfrak{S}_f$  of it there taking place an expression:

$$f(i\alpha) = f(x_i, y_i, z_i, \xi_\alpha, \eta_\alpha, \vartheta_\alpha),$$

and its nondegeneracy meaning the nonzero quality of the two Jacobians:

$$\frac{\partial(f^1(i\alpha), f^2(i\alpha), f^3(i\alpha))}{\partial(x_i, y_i, z_i)} \neq 0, \quad \frac{\partial(f^1(i\alpha), f^2(i\alpha), f^3(i\alpha))}{\partial(\xi_\alpha, \eta_\alpha, \vartheta_\alpha)} \neq 0 \quad (10.14)$$

for dense sets of pairs  $\langle i\alpha \rangle \in \mathfrak{M} \times \mathfrak{N}$ .

The phenomenological symmetry of the trimetric geometry of two sets in question is expressed by the equation

$$\Phi(f(i\alpha), f(i\beta), f(j\alpha), f(j\beta)) = 0, \quad (10.15)$$

in which all the three components of the function  $\Phi = (\Phi_1, \Phi_2, \Phi_3)$  are independent. And that implies that the set of values of the function  $F : \mathfrak{S}_F \rightarrow R^{12}$ , where  $\mathfrak{S}_F \subseteq \mathfrak{M}^3 \times \mathfrak{N}^3$  is its natural domain, belongs locally to the nine-dimensional surface in  $R^{12}$  that is defined by three equations of  $\Phi = 0$ .

Under Theorem 2 of §8 the function (10.13) that gives on the three-dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a trimetric physical structure of rank (2,2) allows a three-dimensional group of motions that consists of two actions of the group  $G^3$  in them. We shall write the actions of that group in  $\mathfrak{M}$  in the explicit form:

$$\left. \begin{aligned} x' &= \lambda(x, y, z; a^1, a^2, a^3), \\ y' &= \sigma(x, y, z; a^1, a^2, a^3), \\ z' &= \tau(x, y, z; a^1, a^2, a^3), \end{aligned} \right\} \quad (10.16)$$

where  $(a^1, a^2, a^3) \in G^3$ . Its action in the other manifold, manifold  $\mathfrak{N}$ , is written similarly:

$$\left. \begin{aligned} \xi' &= \tilde{\lambda}(\xi, \eta, \vartheta; a^1, a^2, a^3), \\ \eta' &= \tilde{\sigma}(\xi, \eta, \vartheta; a^1, a^2, a^3), \\ \vartheta' &= \tilde{\tau}(\xi, \eta, \vartheta; a^1, a^2, a^3), \end{aligned} \right\}$$

the functions  $\tilde{\lambda}, \tilde{\sigma}, \tilde{\tau}$ , which define that action, not necessarily coinciding with the functions  $\lambda, \sigma, \tau$  in the action (10.16). But if those actions are equivalent, then we can always find systems of coordinates in the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  such that  $\lambda = \tilde{\lambda}$ ,  $\sigma = \tilde{\sigma}$ ,  $\tau = \tilde{\tau}$  in them with the corresponding permutation of coordinates in the manifolds.

The invariance of the metric function (10.13) with respect to the group of motions implies its being preserved according to the equation

$$f(x', y', z', \xi', \eta', \vartheta') = f(x, y, z, \xi, \eta, \vartheta), \quad (10.17)$$

which is identically satisfied for each of its components  $f^1, f^2, f^3$  with respect to the coordinates  $x, y, z$  and  $\xi, \eta, \vartheta$  of the points of the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ , as well as to the parameters  $a^1, a^2, a^3$  of the group  $G^3$  acting in them.

**Theorem 3.** *With an accuracy up to a scaling transformation the three-component metric function  $f = (f^1, f^2, f^3)$  that gives on 3-dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a trimetric physical structure (a phenomenologically symmetric geometry of two sets) of rank (2,2) is, in a suitably chosen systems of local coordinates  $x, y, z$  and  $\xi, \eta, \vartheta$  defined by the following canonical expressions:*

$$f^1 = x + \xi, \quad f^2 = y + \eta, \quad f^3 = z + \vartheta; \quad (10.18)$$

$$f^1 = y - \eta, \quad f^2 = (x + \xi)y + z + \vartheta, \quad f^3 = (x + \xi)\eta + z + \vartheta; \quad (10.19)$$

$$\left. \begin{aligned} f^1 &= (x + \xi)^2 \exp(2\frac{y + \eta}{x + \xi}), \\ f^2 &= (x + \xi)z, \quad f^3 = (x + \xi)\vartheta; \end{aligned} \right\} \quad (10.20)$$

$$f^1 = \frac{x + \xi}{y + \eta}, \quad f^2 = (x + \xi)z, \quad f^3 = (x + \xi)\vartheta; \quad (10.21)$$

$$f^1 = (x + \xi)(y + \eta), \quad f^2 = (x + \xi)z, \quad f^3 = (x + \xi)\vartheta; \quad (10.22)$$

$$f^1 = y + \eta, \quad f^2 = (x + \xi)z, \quad f^3 = (x + \xi)\vartheta; \quad (10.23)$$

$$f^1 = \frac{(x + \xi)^p}{y + \eta}, \quad f^2 = (x + \xi)z, \quad f^3 = (x + \xi)\vartheta; \quad (10.24)$$

$$\left. \begin{aligned} f^1 &= (x + \xi)^2 + (y + \eta)^2, \\ f^2 &= z + \arctg \frac{y + \eta}{x + \xi}, \quad f^3 = \vartheta + \arctg \frac{y + \eta}{x + \xi}; \end{aligned} \right\} \quad (10.25)$$

$$\left. \begin{aligned} f^1 &= ((x + \xi)^2 + (y + \eta)^2) \exp(2\gamma \arctg \frac{y + \eta}{x + \xi}), \\ f^2 &= z + \arctg \frac{y + \eta}{x + \xi}, \quad f^3 = \vartheta + \arctg \frac{y + \eta}{x + \xi}; \end{aligned} \right\} \quad (10.26)$$

$$\left. \begin{aligned} f^1 &= \sin y \sin \eta \cos(x + \xi) + \cos y \cos \eta, \\ f^2 &= z + \arcsin \frac{\sin(x + \xi) \sin \eta}{\sqrt{1 - (f^1)^2}}, \\ f^3 &= \vartheta + \arcsin \frac{\sin(x + \xi) \sin y}{\sqrt{1 - (f^1)^2}}; \end{aligned} \right\} \quad (10.27)$$

$$f^1 = (x + \xi)y\eta, \quad f^2 = z + \frac{1}{(x + \xi)y^2}, \quad f^3 = \vartheta + \frac{1}{(x + \xi)\eta^2}, \quad (10.28)$$

where  $0 < |p| < 1$  and  $0 < \gamma < \infty$ .

The proof of the theorem is in §8 of the monograph [24].

Let us now take the equation (10.15) that expresses the phenomenological symmetry of trimetric physical structures of rank (2,2) defined on three-dimensional manifolds by the metric functions (10.18) to (10.28). Just as in the case of the dimetric physical structures (see Theorem 2), we shall make clearer the relation of the metric function (10.13) and the equation (10.15), for which sake we shall use the following theorem proved by R.M. Muradov [32, §19]:

**Theorem 4.** *If a three-component metric function*

$$f = f(x, y, z, \xi, \eta, \vartheta)$$

*gives on 3-dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a trimetric physical structure of rank (2,2), then, with an accuracy up to a scaling transformation and a change of coordinates in the manifolds, it defines in  $R^3$  a quazigroup operation with a right identity and its right inverse that coincides with the parent element such that in the equation expressing the phenomenological symmetry of the respective geometry of two sets under the sign of the operator of alternation  $\hat{R}(\alpha\beta)$  there stands an expression similar to the metric function itself and derived from it under the substitutions of  $x \rightarrow f^1(i\alpha)$ ,  $y \rightarrow f^2(i\alpha)$ ,  $z \rightarrow f^3(i\alpha)$ ,  $\xi \rightarrow f^1(j\alpha)$ ,  $\eta \rightarrow f^2(j\alpha)$ ,  $\vartheta \rightarrow f^3(j\alpha)$ :*

$$\hat{R}(\alpha\beta)f(f^1(i\alpha), f^2(i\alpha), f^3(i\alpha), f^1(j\alpha), f^2(j\alpha), f^3(j\alpha)) = 0;$$

$$f^1 = x - \xi, \quad f^2 = y - \eta, \quad f^3 = z - \vartheta; \quad (10.18')$$

$$f^1 = x - \xi, \quad f^2 = y - \eta, \quad f^3 = (x - \xi)\eta + z - \vartheta; \quad (10.19')$$

$$f^1 = (x - \xi)\vartheta, \quad f^2 = (y - \eta - (x - \xi) \ln \vartheta)\vartheta, \quad f^3 = z/\vartheta; \quad (10.20')$$

$$f^1 = (x - \xi)\vartheta, \quad f^2 = (y - \eta)\vartheta, \quad f^3 = z/\vartheta; \quad (10.21')$$

$$f^1 = (x - \xi)\vartheta, \quad f^2 = (y - \eta)/\vartheta, \quad f^3 = z/\vartheta; \quad (10.22')$$

$$f^1 = (x - \xi)\vartheta, \quad f^2 = y - \eta, \quad f^3 = z/\vartheta; \quad (10.23')$$

$$f^1 = (x - \xi)\vartheta, \quad f^2 = (y - \eta)\vartheta^p, \quad f^3 = z/\vartheta; \quad (10.24')$$

where  $0 < |p| < 1$ ;

$$\left. \begin{aligned} f^1 &= (x - \xi) \cos \vartheta - (y - \eta) \sin \vartheta, \\ f^2 &= (x - \xi) \sin \vartheta + (y - \eta) \cos \vartheta, \\ f^3 &= z - \vartheta; \end{aligned} \right\} \quad (10.25')$$

$$\left. \begin{aligned} f^1 &= \frac{(x - \xi) \cos \vartheta - (y - \eta) \sin \vartheta}{\exp(\gamma \vartheta)}, \\ f^2 &= \frac{(x - \xi) \sin \vartheta + (y - \eta) \cos \vartheta}{\exp(\gamma \vartheta)}, \\ f^3 &= z - \vartheta; \end{aligned} \right\} \quad (10.26')$$

where  $0 < \gamma < \infty$ ;

$$\left. \begin{aligned} f^1 &= x\sqrt{1 - \xi^2 - \eta^2 - \vartheta^2} - \xi\sqrt{1 - x^2 - y^2 - z^2} + y\vartheta - z\eta, \\ f^2 &= y\sqrt{1 - \xi^2 - \eta^2 - \vartheta^2} - \eta\sqrt{1 - x^2 - y^2 - z^2} + z\xi - x\vartheta, \\ f^3 &= z\sqrt{1 - \xi^2 - \eta^2 - \vartheta^2} - \vartheta\sqrt{1 - x^2 - y^2 - z^2} + x\eta - y\xi; \end{aligned} \right\} \quad (10.27')$$

$$\left. \begin{aligned} f^1 &= \frac{(x - \xi)\eta^2}{1 - (x - \xi)\vartheta\eta^2}, \\ f^2 &= \frac{(1 - (x - \xi)\vartheta\eta^2)y}{\eta}, \\ f^3 &= z - \frac{\vartheta\eta^2}{(1 - (x - \xi)\vartheta\eta^2)y^2}. \end{aligned} \right\} \quad (10.28')$$

We shall note in conclusion that V.A. Kyrov, in his note [18], simultaneously with the classification of the four-metric phenomenologically symmetric geometries of rank 3 on one set, which is at the end of §3, built the classification of the four-metric physical structures (phenomenologically symmetric geometries of two sets) of rank (2,2).

### §11. The group symmetry of arbitrary physical structures

Binary physical structures as phenomenologically symmetric geometries are naturally defined on one and two sets. A two-point function that defines such a geometry allows a nontrivial group of motions with a finite number of continuous parameters which number we named the degree of the group symmetry. Under certain relationship among the rank of the physical structure, the number of the essential parameters of the group of motions, and the dimensionality of the sets, the group and phenomenological symmetries of the respective geometry turn out to be equivalent. Those relationships were

built into the definition of the physical structure and the phenomenological and group symmetries of it. The question naturally suggests itself of their origin and rationale. Besides, a good many potentialities spring up for generalization and development of the notion of a physical structure, one of which was realized in §1 and in §8, when two points were assigned more than one real number. Another potentiality of generalization, realized in §5, consists in defining the ternary physical structures, i.e. such that the metric function defining the physical structure does not assigns a number to two but to three points. The ternary physical structures are naturally defined on one, two, and three sets, and the cases of their minimal rank are treated by the author in his notes [5], [33], and [34]. However, as early as on the stage of the preliminary investigation it was already found that the ternary structures, in contrast to the binary ones, are not endowed with a group symmetry, i.e. the initial three-point function does not allow a nontrivial group of motions. Such a result throws into relief the rich potentialities of the binary structures, which are in such a contrast with the ternary ones, which only exist in case of the smallest possible rank. So a question also suggests itself of the intrinsic causes of such a difference.

To give the final answer to the question of the relationship of the rank of a physical structure, the degree of the group symmetry, and the dimensionality of the sets (manifolds) where the structure is defined, as well as the question of the difference between binary and polyary (ternary, in particular) structures, it is necessary to proceed from a more general definition of a physical structure. Then we shall be able to establish what are the relations among the principle characteristics of a structure that make it able to be endowed with a group symmetry, and what are those that render it unable to be so endowed. It is natural to suppose that only structures whose groups of motions are nontrivial may have real physical and mathematical meaning. For the sake of brevity of the further exposition, the definition of arbitrary physical structures will be given in the most general form, sufficient, however, for the exposition to be cogent.

Suppose there are  $p$  sets  $\mathfrak{M}_1, \dots, \mathfrak{M}_p$  of arbitrary nature, each being, mathematically, a smooth manifold of dimension  $m_1, \dots, m_p$ , respectively.



Suppose there is also a function

$$f : \mathfrak{S}_f \rightarrow R^s, \quad (11.1)$$

where  $\mathfrak{S}_f \subseteq \mathfrak{M}_1^{q_1} \times \dots \times \mathfrak{M}_p^{q_p}$  that assigns to each cortege of length  $q = q_1 + \dots + q_p$  from  $\mathfrak{S}_f$  a point from  $R^s$ , i.e.  $s$  real numbers. It is supposed that the domain  $\mathfrak{S}_f$  of the function  $f$  is open and dense in  $q$ -ary direct product  $\mathfrak{M}_1^{q_1} \times \dots \times \mathfrak{M}_p^{q_p}$  of the assumed sets  $\mathfrak{M}_1, \dots, \mathfrak{M}_p$ , and its coordinate representation is sufficiently smooth. We shall call the numerical cortege  $(q_1, \dots, q_p)$  a *multiplicity*, the number  $q = q_1 + \dots + q_p$  – an *arity*, and the function (11.1) a *metric* one.

Let  $M_1, \dots, M_p$  be arbitrary integers, such that  $M_1 > q_1, \dots, M_p > q_p$ . We shall build a mapping

$$F : \mathfrak{S}_F \rightarrow R^{sC_{M_1}^{q_1} \times \dots \times C_{M_p}^{q_p}}, \quad (11.2)$$

where  $\mathfrak{S}_F \subseteq \mathfrak{M}_1^{M_1} \times \dots \times \mathfrak{M}_p^{M_p}$ , by assigning to every cortege of length  $M_1 + \dots + M_p$  from  $\mathfrak{S}_F$  a collection ordered with respect to it of  $sC_{M_1}^{q_1} \times \dots \times C_{M_p}^{q_p}$  numbers corresponding all the corteges of length  $q = q_1 + \dots + q_p$  which are the projections of the initial cortege onto the region  $\mathfrak{S}_f$ . The domain  $\mathfrak{S}_F$  of the function (11.2) will obviously be open and dense in  $\mathfrak{M}_1^{M_1} \times \dots \times \mathfrak{M}_p^{M_p}$ . Quite similarly, we shall build another mapping

$$F' : \mathfrak{S}_{F'} \rightarrow R^{sC_{M'_1}^{q_1} \times \dots \times C_{M'_p}^{q_p}}, \quad (11.2')$$

where  $\mathfrak{S}_{F'} \subseteq \mathfrak{M}_1^{M'_1} \times \dots \times \mathfrak{M}_p^{M'_p}$  and  $M'_1 \geq M_1, \dots, M'_p \geq M_p$ . The projection of the mapping  $F'$  may be obtained by way of dropping a collection of corteges of length  $q = q_1 + \dots + q_p$  from its domain  $\mathfrak{S}_{F'}$ , together with dropping all the numbers that correspond those corteges with respect to the function (11.1) from its range of values.

**Definition 1.** We shall say that the function (11.1) gives on  $m_1, \dots, m_p$ -dimensional manifolds  $\mathfrak{M}_1, \dots, \mathfrak{M}_p$  a  $q$ -ary polymetric physical structure of rank  $(M_1, \dots, M_p)$  and of arity  $(q_1, \dots, q_p)$ , if on the set dense in  $\mathfrak{S}_F$  the rank of the mapping  $F$  is equal to  $s(C_{M_1}^{q_1} \times \dots \times C_{M_p}^{q_p} - 1)$ , and the rank of any projection of the mapping  $F'$  whose domain does not include any region of the mapping  $F$  is maximal on a set dense in  $\mathfrak{S}_{F'}$ .

In other words, the set of values of the mapping  $F$  is locally a subset of the set of zeros of the system of  $s$  independent functions of  $\Phi = (\Phi_1, \dots, \Phi_s)$

of  $sC_{M_1}^{q_1} \times \dots \times C_{M_p}^{q_p}$  variables,  $s$  functional relations

$$\Phi = (\Phi_1, \dots, \Phi_s) = 0 \quad (11.3)$$

being generating ones, in the sense that any other nontrivial relations will be nothing more than their corollaries.

**Definition 2.** We shall say that a physical structure we have so defined is endowed with a group symmetry of finite degree  $r \geq 1$  if there are also effective smooth local actions defined of some  $r$ -dimensional local Lie group  $G^r$  in the manifolds  $\mathfrak{M}_1, \dots, \mathfrak{M}_p$ , such that for their mutual expansion onto the direct product  $\mathfrak{M}_1^{q_1} \times \dots \times \mathfrak{M}_p^{q_p}$  the metric function (11.1) is a  $q$ -point invariant,  $q$  being  $q_1 + \dots + q_p$ .

Since the manifolds being transformed are finite dimensional, the condition in Definition 2 of the maximal number  $r$  of the continuous parameters of the group of motions being also finite is quite natural, and the group itself is therefore a finite dimensional local Lie group of special transformations of the manifold  $\mathfrak{M}_1^{q_1} \times \dots \times \mathfrak{M}_p^{q_p}$  of dimension  $q_1 m_1 + \dots + q_p m_p$  which transformations are a mutual expansion of the transformations of the manifolds  $\mathfrak{M}_1, \dots, \mathfrak{M}_p$ .

Let us write a system of  $sC_{M'_1}^{q_1} \times \dots \times C_{M'_p}^{q_p}$  equations of conservation of the metric function (11.1):

$$Df|_{F'} = 0 \quad (11.4)$$

with respect to  $M'_1 m_1 + \dots + M'_p m_p$  differentials of the coordinates of the points of a cortege from  $\mathfrak{S}_{F'}$ . In case the physical structure is endowed with a group symmetry, the homogeneous system (11.4) must, on the one hand, have at least one nonzero solution, and the number of its linearly independent nonzero solutions for any numbers  $M'_1, \dots, M'_p$  must not, on the other hand, be bigger than some finite value equal to the degree of the group symmetry. It is known that the number of such solutions is equal to the number of the unknowns in the system minus the rank of the matrix. But the matrix of the system of equations (11.4) is a functional matrix for the system of functions  $f$  which correspond all the projections of the domain  $\mathfrak{S}_{F'}$  of the mapping (11.2') onto the domain  $\mathfrak{S}_f$  of the assumed function (11.1). Obviously, the rank of the matrix will not change if we eliminate from the system of functions  $f|_{F'}$  the functions dependent with respect to

the relation (11.3). Their elimination yields the maximal projection of the mapping (11.2'), which projection does not contain the mapping (11.2). We shall designate the number of functions  $f$  in that maximal projection by  $N(M'_1, \dots, M'_p)$ . Then, under Definition 1 of a physical structure the rank of the matrix of the system of equations (11.4) will be equal to

$$\min(M'_1 m_1 + \dots + M'_p m_p; N(M'_1, \dots, M'_p)).$$

If values of the numbers  $M'_1, \dots, M'_p$ , may be found such that  $M'_1 m_1 + \dots + M'_p m_p \leq N(M'_1, \dots, M'_p)$ , then the rank of the matrix of the system of equations (11.4) for them will be equal to  $M'_1 m_1 + \dots + M'_p m_p$ , i.e. to the number of the unknowns in the matrix. But then the system may only have a zero solution which means the absence of a nontrivial group symmetry of the physical structure in question. And in case for any values of  $M'_1, \dots, M'_p$  a strict inequality  $N(M'_1, \dots, M'_p) < M'_1 m_1 + \dots + M'_p m_p$  takes place the rank of the matrix of the system (11.4) is equal to  $N(M'_1, \dots, M'_p)$  and the number of its linearly independent zero solutions turns out to be

$$r' = M'_1 m_1 + \dots + M'_p m_p - N(M'_1, \dots, M'_p) > 0.$$

As mentioned above, the number  $r'$  must not exceed some finite value if a physical structure is endowed with a group symmetry under Definition 2. Let us establish at which correlations of the dimensionality of the sets the arity and the rank the physical structure defined by the metric function (11.1) thus restricted may have a group symmetry and let us find the degree  $r$  of that symmetry.

Binary physical structures ( $q = 2$ ) may be defined by the function (11.1) on one or two sets. The case of one set (that of  $p=1$ ), where we named them binary phenomenologically symmetric geometries, was discussed in detail in §5. The relationships (5.7) and (5.8) we deduced in §5 established the relation between the dimension of the manifold on the one hand and the rank of the phenomenological symmetry and the degree of the group symmetry on the other. Physical structures were defined as phenomenologically symmetric geometries of two sets (those with  $p=2$ ), and we are going to discuss them now, whereby we are going to add more detail to what was said in §5.

Binary physical structure of rank  $(M, N)$  and arity  $(1, 1)$  where  $M \geq 2$  and  $N \geq 2$  is defined by the metric function (11.1) on two manifolds,  $\mathfrak{M}$  and  $\mathfrak{N}$  of dimensions  $m$  and  $n$  respectively where  $\mathfrak{S}_f \subseteq \mathfrak{M} \times \mathfrak{N}$ . Under Definition 1, the rank of the mapping  $F : \mathfrak{S}_F \rightarrow R^{sMN}$ , where  $\mathfrak{S}_F \subseteq \mathfrak{M}^M \times \mathfrak{N}^N$ , is equal to  $s(MN - 1)$ . The number of the dependents in the system of  $sM'N'$  functions of the mapping  $F' : \mathfrak{S}_{F'} \rightarrow R^{sM'N'}$ , where  $\mathfrak{S}_{F'} \subseteq \mathfrak{M}^{M'} \times \mathfrak{N}^{N'}$  and  $M' \geq M$ ,  $N' \geq N$ , is determined by way of superposing the matrix of pairs for the cortege of length  $M + N$  from the region  $\mathfrak{S}_F$  on the matrix of pairs for the cortege of length  $M' + N'$  from the region  $\mathfrak{S}_{F'}$ , in a way similar to that described in §5. It is very easy to find that it is equal to  $s(M' - M + 1)(N' - N + 1)$ . So, the rank of the system of functions  $f|_{F'}$ , and of the system of equations (11.4) is according to Definition 1 is equal to

$$\min(M'm + N'n; sM'N' - s(M' - M + 1)(N' - N + 1)).$$

If  $m < s(N - 1)$  or  $n < s(M - 1)$ , then for some values of  $M'$  and  $N'$  the rank of the matrix of the system of equations (11.4) will be equal to the number of the unknowns  $M'm + N'n$  in it, and for them it only has a zero solution, which implies the absence of any group symmetry of the physical structure in question. And if  $m \geq s(N - 1)$  and  $n \geq s(M - 1)$ , then the rank of the matrix of the system of equations (11.4) for any values  $M'$  and  $N'$  is equal to  $sM'N' - s(M' - M + 1)(N' - N + 1)$ , and so it has

$$r' = M'm + N'n - sM'(N - 1) - sN'(M - 1) + s(M - 1)(N - 1)$$

linearly independent nonzero solutions. With either  $m > s(N - 1)$  or  $n > s(M - 1)$ , increasing  $M'$  and  $N'$  may result in the number  $r'$  of such solutions becoming arbitrarily large, which is in contradiction with the assumption of the finiteness of the degree of the group. That's why the binary physical structure of rank  $(M, N)$  defined on an  $m$ -dimensional and an  $n$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  by the metric function (11.1) is only endowed with a group symmetry if the conditions are satisfied as follows:

$$\mathbf{m} = \mathbf{s}(\mathbf{N} - \mathbf{1}), \quad \mathbf{n} = \mathbf{s}(\mathbf{M} - \mathbf{1}). \quad (11.5)$$

The degree  $r$  of the group symmetry, i.e. the number of independent and essential parameters of the group of motions, is equal to the number  $r'$  of

the linearly independent nonzero solutions of the system of equations (11.4) with the relations (11.5):

$$\mathbf{r} = \mathbf{s}(\mathbf{M} - \mathbf{1})(\mathbf{N} - \mathbf{1}) = \mathbf{mn}/\mathbf{s}. \quad (11.6)$$

The relations (11.5) and (11.6) were used in the principle definitions of the author's note [35] and in §8 of this monograph. It is only necessary to keep in mind some small change of designation:  $M \rightarrow n + 1$ ,  $N \rightarrow m + 1$ ,  $m \rightarrow sm$ ,  $n \rightarrow sn$ . That is, on an  $sm$ -dimensional and an  $sn$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  the metric function (11.1) gives a physical structure of rank  $(n + 1, m + 1)$  endowed with a group symmetry of degree  $r = smn$ .

The ternary physical structures (with  $q=3$ ) may be defined by the function (11.1) on one, two, or three sets. The case of one set ( $p=1$ ) was already discussed in §5, where they were called ternary phenomenologically symmetric geometries, where they turned out unable to have a group symmetry of finite degree. It is natural to expect that we shall have the same result in case of two ( $p=2$ ) and three ( $p=3$ ) sets. Let us establish that.

Ternary physical structures of rank  $(M, N)$  and multiplicity  $(2,1)$ , where  $M \geq 3$ ,  $N \geq 2$ , are defined on an  $m$ -dimensional and an  $n$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  by the metric function (11.1), where  $\mathfrak{S}_f \subseteq \mathfrak{M}^2 \times \mathfrak{N}$ . Under Definition 1, the rank of the mapping  $F : \mathfrak{S}_F \rightarrow R^{sM(M-1)N/2}$ , where  $\mathfrak{S}_F \subseteq \mathfrak{M}^M \times \mathfrak{N}^N$ , is equal to  $sM(M-1)N/2 - s$ . The rank of the mapping  $F' : \mathfrak{S}_{F'} \rightarrow R^{sM'(M'-1)N'/2}$ , where  $\mathfrak{S}_{F'} \subseteq \mathfrak{M}^{M'} \times \mathfrak{N}^{N'}$  and  $M' \geq M$ ,  $N' \geq N$ , that is the rank of the matrix of the system of equations (11.4), may be found by way of superposition of the matrix of triples for a cortege of length  $M + N$  from the region  $\mathfrak{S}_F$  on the matrix of triples for a cortege of length  $M' + N'$  from the region  $\mathfrak{S}_{F'}$ :

$$\begin{aligned} & \min(M'm + N'n; sM'(M' - 1)N'/2 - \\ & - s(M' - M + 1)(M' - M + 2)(N' - N + 1)/2). \end{aligned}$$

Since  $M > 2$  and  $N > 1$ , for sufficiently big values of  $M'$  and  $N'$  that rank is equal to  $M'm + N'n$ , i.e. to the number of the unknowns in the system of equations (11.4), which system may have only a nonzero solution. Thus, the ternary physical structures on two sets may not be endowed with a group

symmetry of finite degree.

For the ternary physical structure of rank  $(M, N, L)$  and multiplicity  $(1, 1, 1)$ , where  $M \geq 2$ ,  $N \geq 2$ ,  $L \geq 2$ , defined on three manifolds  $\mathfrak{M}, \mathfrak{N}, \mathfrak{L}$  of dimensions  $m, n, l$  respectively by the metric function (11.1), where  $\mathfrak{S}_f \subseteq \mathfrak{M} \times \mathfrak{N} \times \mathfrak{L}$ , the rank of the mapping  $F : \mathfrak{S}_F \rightarrow R^{sMNL}$ , where  $\mathfrak{S}_F \subseteq \mathfrak{M}^M \times \mathfrak{N}^N \times \mathfrak{L}^L$ , under Definition 1, is equal to  $sMNL - s$ . And the rank of the mapping  $F' : \mathfrak{S}_{F'} \rightarrow R^{sM'N'L'}$ , where  $\mathfrak{S}_{F'} \subseteq \mathfrak{M}^{M'} \times \mathfrak{N}^{N'} \times \mathfrak{L}^{L'}$  and  $M' \geq M$ ,  $N' \geq N$ ,  $L' \geq L$ , i.e. the rank of the matrix of equations (11.4), is as easily found by the same method of superposition:

$$\begin{aligned} & \min(M'm + N'n + L'l; sM'N'L' - \\ & - s(M' - M + 1)(N' - N + 1)(L' - L + 1)). \end{aligned}$$

Since  $M > 1$ ,  $N > 1$ ,  $L > 1$ , the rank of the mapping  $F'$  for sufficiently great values of  $M', N', L'$  is equal to  $M'm + N'n + L'l$ , i.e. to the number of the unknowns in the system of equations (11.4), which system for them has only a nonzero solution. Thus, ternary physical structures cannot have a group symmetry of finite degree on three sets either.

The physical structures of rank  $(M_1, \dots, M_p)$  and multiplicity  $(q_1, \dots, q_p)$ , where  $M_1 > q_1, \dots, M_p > q_p$ , are defined on  $p$  manifolds  $\mathfrak{M}_1, \dots, \mathfrak{M}_p$  of dimensions  $m_1, \dots, m_p$  respectively by the metric function (11.1). The rank of the mapping (11.2) is, under Definition 1, equal to  $s(C_{M_1}^{q_1} \times \dots \times C_{M_p}^{q_p} - 1)$ , and the rank of the mapping (11.2'), i.e. the rank of the system of equations (11.4), may be found by superposing on the matrix of corteges of length  $q = q_1 + \dots + q_p$  for a cortege of length  $M'_1 + \dots + M'_p$  from the region  $\mathfrak{S}_{F'}$  that of corteges of the same length  $q$  for a cortege of length  $M_1 + \dots + M_p$  from the region  $\mathfrak{S}_F$ :

$$\begin{aligned} & \min(M'_1 m_1 + \dots + M'_p m_p; sC_{M'_1}^{q_1} \times \dots \times C_{M'_p}^{q_p} - \\ & - sC_{M'_1 - M_1 + q_1}^{q_1} \times \dots \times C_{M'_p - M_p + q_p}^{q_p}). \end{aligned}$$

As binary ( $q = 2$ ) and ternary ( $q = 3$ ) physical structures have been investigated, we shall assume that their arity  $q > 3$ . The number of the unknowns in the system of equations (11.4) linearly depends on  $M'_1, \dots, M'_p$ . At the same time, the residual that is part of the second half of the latter expression contains, with respect to the same variables  $M'_1, \dots, M'_p$ , members

of order  $q - 1 > 2$  whose number becomes unrestrictedly large because  $M_1 > q_1, \dots, M_p > q_p$ . And that means that for sufficiently great values of  $M'_1, \dots, M'_p$  the rank of the matrix of the system of equations (11.4) will become equal to the number of the unknowns in it, and therefore it will only have a zero solution for them. Thus,  $q$ -ary physical structures defined on sets  $\mathfrak{M}_1, \dots, \mathfrak{M}_p$  by the function (11.1) cannot be endowed with a group symmetry of finite degree in the case of  $q > 3$  either.

The final conclusion we arrive at under the results of the exposition above is expressed by the following theorem.

**Theorem.** *It is only binary physical structures on one or two sets that can be endowed with a group symmetry of finite degree, while for  $q$ -ary physical structures with  $q \geq 3$  the metric function (11.1) does not allow any nontrivial local motions.*

The group symmetry of binary physical structures, which were the principle object of investigation in monographs [10] and [24] by the author, is the determining one. That is, the function  $f : \mathfrak{S}_f \rightarrow R^s$  where either  $\mathfrak{S}_f \subseteq \mathfrak{M}^2$  or  $\mathfrak{S}_f \subseteq \mathfrak{M} \times N$  defines a physical structure if and only if it allows a nontrivial finite dimensional group of motions. The condition of physical structures having a group symmetry of finite degree determines that degree, establishing whereby its relationship with the dimensionality of the sets and the rank of the structure by the relations (5.7), (5.8) and (11.5), (11.6). On the other hand, without the assumption of a group symmetry of finite degree even the relations (5.7) and (11.5), establishing the relationship of the dimensionality of the sets with the rank of the structure and containing no degree of a group symmetry, must be additionally stipulated in the initial axioms without any sufficient validation of the presence of any such relation.

We shall note in conclusion that the results of this paragraph were published by the author in his note [36].

## §12. Functional equations in the theory of physical structures

In the mathematical apparatus of the theory of physical structures (TPS) functional equations are of key importance, and it is worth noting that the phenomenological and group symmetry yield different types of such equations. In this paragraph we are going to discuss the functional equations for physical structures on two sets. We remind that for geometric physical structures on one set the respective functional equations were discussed in §6.

The physical structure of rank  $(n+1, m+1)$  is defined by the nondegenerate  $s$ -component metric function

$$f = f(x, \xi) = f(x^1, \dots, x^{sm}, \xi^1, \dots, \xi^{sn}), \quad (12.1)$$

where  $m, n, s \geq 1$ , on an  $sm$ - and an  $sn$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ . Its phenomenological symmetry means that for every cortege  $\langle ijk \dots v, \alpha\beta\gamma \dots \tau \rangle$  from some neighbourhood  $U \subset \mathfrak{S}_F \subseteq \mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1}$  of a set of corteges dense in  $\mathfrak{S}_F$  the identity is satisfied as follows:

$$\Phi(f(i\alpha), f(i\beta), \dots, f(v\tau)) = 0, \quad (12.2)$$

, in which the function  $\Phi$ , as well as the metric function (12.1), has  $s$  components that are independent.

The identity (12.2) is, on the one hand, an analytical expression of the principle of phenomenological symmetry, and on the other, is a functional equation in the TPS. In the general case, the unknowns in the equation (12.2) are both the function  $f = (f^1, \dots, f^s)$  that gives the physical structure and the function  $\Phi = (\Phi_1, \dots, \Phi_s)$  that expresses via that equation the phenomenological symmetry of the physical structure. Understood in that sense, the functional equation (12.2) has been solved only for some particular cases. It is solved completely for  $s = 1$  (see §9), and partly for  $s = 2$  (see §10) and  $s = 3, 4$  (see §10). A simpler variant of the equation (12.2) is that with one of the functions  $f$  and  $\Phi$  known. More often it is the function  $f$  that is known, so that is the case we take into consideration first.

The simplest cases of the equation (12.2) with the known function  $f$  take place in the analysis of the Second law of Newton and Ohm's law. In the



traditional Newtonian law

$$a = F/m, \quad (12.3)$$

where  $a$  is a function of the acceleration of a body of weight  $m$  under the impact of the accelerator  $F$ , it is easy to find for any two bodies  $i, j$  and any two accelerators  $\alpha, \beta$  the relation that only comprises the four accelerations measured in experiment:

$$a_{i\alpha}a_{j\beta} - a_{i\beta}a_{j\alpha} = 0. \quad (12.4)$$

The phenomenologically symmetric form (12.4) of Newton's 2nd law is the solution of the functional equation

$$\Phi(a_{i\alpha}, a_{i\beta}, a_{j\alpha}, a_{j\beta}) = 0, \quad (12.5)$$

in which the function of acceleration  $a$  is known from the ordinary formula (12.3) of the law. The function  $a$  gives on the set of material bodies  $\mathfrak{M}$  and the set of accelerators  $\mathfrak{N}$  a physical structure of minimal rank (2,2), and so the equation (12.5) turns out to be simple enough. It is quite obvious that that equation is a special case of the general equation (12.2) for the case of  $m = n = s = 1$  if we set  $f = a$  in it.

Now let us take Ohm's law:

$$I = \mathcal{E}/(R + r), \quad (12.6)$$

where  $I$  is the function of current measured by ammeter in a closed circuit that contains a conductor with resistance  $R$  and a source of current that has electromotive force  $\mathcal{E}$  and internal resistance  $r$ . For any three conductors  $i, j, k$  and any two sources of current  $\alpha, \beta$ , the six possible values of current are tied by the relationship that does not contain the characteristics of theirs:

$$\begin{vmatrix} (I_{i\alpha})^{-1} & (I_{i\beta})^{-1} & 1 \\ (I_{j\alpha})^{-1} & (I_{j\beta})^{-1} & 1 \\ (I_{k\alpha})^{-1} & (I_{k\beta})^{-1} & 1 \end{vmatrix} = 0. \quad (12.7)$$

The relation (12.7) defines the phenomenologically symmetric form of Ohm's law that is the solution of the functional equation

$$\Phi(I_{i\alpha}, I_{i\beta}, I_{j\alpha}, I_{j\beta}, I_{k\alpha}, I_{k\beta}) = 0, \quad (12.8)$$

in which the function of the current  $I$  is known from the ordinary form (12.6) of the same law. We shall note that the function  $I$  gives on the set of conductors  $\mathfrak{M}$  and the set of current sources  $\mathfrak{N}$  a physical structure of rank (3,2), and the equation (12.8) is derived from the general equation (12.2) if  $n = 2, m = s = 1$ , and if we set  $f = I$  in it.

In §9 we gave all the possible expressions (9.2)–(9.7) of the function  $f$  giving on an  $m$ -dimensional and an  $n$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a unimetric physical structure of rank  $(n+1, m+1)$ . For each of them, except perhaps for the expression (9.4), the solution of the functional equation (12.2) is relatively easy to find and it is done in a purely algebraic way at that: by way of excluding from the  $(m+1)(n+1)$  values of the function  $f$  that correspond to all the pairs of points of the cortege  $\langle ijk \dots v, \alpha\beta\gamma \dots \dots \tau \rangle \in \mathfrak{S}_F \subseteq \mathfrak{M}^{n+1} \times \mathfrak{N}^{m+1}$  the coordinates of the points of that cortege. The relations (9.2')–(9.7') that that artifice yields are the respective solutions of the functional equation (12.2).

In §10 we gave all the possible expressions (10.3)–(10.12) for the two-component function  $f = (f^1, f^2)$  that defines on a two-dimensional and a  $2n$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a physical structure of rank  $(n+1, 2)$ , where  $n \geq 1$ . The functional equation (12.2) for the dimetric physical structure of rank  $(n+1, 2)$  is as follows:

$$\Phi(f(i\alpha), f(i\beta), f(j\alpha), f(j\beta), \dots, f(v\alpha), f(v\beta)) = 0, \quad (12.9)$$

and we are to keep in mind that the functions  $f$  and  $\Phi$  are two-component ones, i.e.  $f = (f^1, f^2)$  and  $\Phi = (\Phi_1, \Phi_2)$ . The solutions of that equation for each of the functions (10.3)–(10.12) are the expressions that follow Theorem 1 of §10. Theorem 2 determines the solutions of the same functional equation (12.9), but with different equivalent expressions (10.3')–(10.12') for the metric function.

§10 also gives the complete classification (10.18)–(10.28) of the three-component functions  $f = (f^1, f^2, f^3)$  that give on three-dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a physical structure of rank (2,2). The functional equation (12.2) for them is as follows:

$$\Phi(f(i\alpha), f(i\beta), f(j\alpha), f(j\beta)) = 0, \quad (12.10)$$

which is a system of three functional equations, as the function  $\Phi = (\Phi_1, \Phi_2, \Phi_3)$

it contains is also a three-component one. The solutions of the equation (12.10) for the equivalent expressions (10.18')–(10.28') are determined by Theorem 4 of §10.

Another case, namely that of the functional equation (12.2) where the function  $\Phi = (\Phi_1, \dots, \Phi_s)$  is known and the metric function  $f = (f^1, \dots, f^s)$  defining on the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  of dimensions  $sm$  and  $sn$  a physical structure of rank  $(n+1, m+1)$  is to be found is solved as follows. In the direct products  $\mathfrak{M}^n$  and  $\mathfrak{N}^m$  corteges  $\langle j_0 k_0 \dots v_0 \rangle$  and  $\langle \beta_0 \gamma_0 \dots \tau_0 \rangle$  are fixed of lengths  $n$  and  $m$  respectively. Points of the corteges are selected such that the equation (12.2) written for the cortege  $\langle ij_0 k_0 \dots v_0, \alpha \beta_0 \gamma_0 \dots \tau_0 \rangle \in \mathfrak{S}_F$  might be solved uniquely with respect to  $f(i\alpha)$ . Next, local coordinates  $x_i^1, x_i^2, \dots, x_i^{sm}$  and  $\xi_\alpha^1, \xi_\alpha^2, \dots, \xi_\alpha^{sn}$ , are introduced in a suitable way, and it is via them that the metric function (12.1) is expressed. If the expression obtained is nondegenerate and its substitution into the initial equation (12.2) yields an identity with respect to all the coordinates of the points of the cortege  $\langle ijk \dots v, \alpha \beta \gamma \dots \tau \rangle$ , then the metric function found does give a physical structure of rank  $(n+1, m+1)$ .

We shall illustrate the method we have described of solving the functional equation (12.2) with an example of a unimetric and dimetric physical structures of minimal rank (2,2).

We shall write the phenomenologically symmetric relation

$$f(i\alpha) - f(i\beta) - f(j\alpha) + f(j\beta) = 0$$

for a quadruple  $\langle ij_0, \alpha\beta_0 \rangle$ :

$$f(i\alpha) - f(i\beta_0) - f(j_0\alpha) + f(j_0\beta_0) = 0,$$

and then solve it with respect to  $f(i\alpha)$ :

$$f(i\alpha) = f(i\beta_0) + f(j_0\alpha) - f(j_0\beta_0).$$

By introducing coordinates  $x_i = f(i\beta_0) - f(j_0\beta_0)/2$  and  $\xi_\alpha = f(j_0\alpha) - f(j_0\beta_0)/2$ , where  $f(j_0\beta_0)$  is, obviously, a constant, we get the coordinate representation of the metric function (9.2) that, substituted into the initial relation, turns it into an identity confirming whereby that that function does give on one-dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a unimetric physical structure of rank (2,2).

Let us take a more complicated phenomenologically symmetric relation

$$\left. \begin{aligned} \left| \begin{array}{cc} f^1(i\alpha) - f^1(i\beta) & f^1(i\alpha)f^2(j\alpha) \\ f^1(j\alpha) - f^1(j\beta) & f^1(j\alpha)f^2(i\alpha) \end{array} \right| &= 0, \\ \left| \begin{array}{cc} f^2(i\alpha) - f^2(j\alpha) & f^2(i\alpha)f^1(i\beta) \\ f^2(i\beta) - f^2(j\beta) & f^2(i\beta)f^1(i\alpha) \end{array} \right| &= 0, \end{aligned} \right\}$$

which we shall consider as a system of two functional equations, and shall write them first for the quadruple  $\langle j_0 i, \beta_0 \alpha \rangle$ :

$$\left. \begin{aligned} \left| \begin{array}{cc} f^1(j_0\beta_0) - f^1(j_0\alpha) & f^1(j_0\beta_0)f^2(i\beta_0) \\ f^1(i\beta_0) - f^1(i\alpha) & f^1(i\beta_0)f^2(j_0\beta_0) \end{array} \right| &= 0, \\ \left| \begin{array}{cc} f^2(j_0\beta_0) - f^2(i\beta_0) & f^2(j_0\beta_0)f^1(j_0\alpha) \\ f^2(j_0\alpha) - f^2(i\alpha) & f^2(j_0\alpha)f^1(j_0\beta_0) \end{array} \right| &= 0, \end{aligned} \right\}$$

solving them whereafter with respect to  $f(i\alpha) = (f^1(i\alpha), f^2(i\alpha))$ :

$$\left. \begin{aligned} f^1(i\alpha) &= [f^2(i\beta_0)f^1(j_0\beta_0) + f^1(j_0\alpha)f^2(j_0\beta_0) - \\ &\quad - f^1(j_0\beta_0)f^2(j_0\beta_0)]f^1(i\beta_0)/f^2(i\beta_0)f^1(j_0\beta_0), \\ f^2(i\alpha) &= [f^2(i\beta_0)f^1(j_0\beta_0) + f^1(j_0\alpha)f^2(j_0\beta_0) - \\ &\quad - f^1(j_0\beta_0)f^2(j_0\beta_0)]f^2(j_0\alpha)/f^1(j_0\alpha)f^2(j_0\beta_0). \end{aligned} \right\}$$

By way of introducing suitable coordinates  $x_i = f^2(i\beta_0)f^1(j_0\beta_0)$ ,  $y_i = f^1(i\beta_0)/f^2(i\beta_0) \times f^1(j_0\beta_0)$  and  $\xi_\alpha = (f^1(j_0\alpha) - f^1(j_0\beta_0))f^2(j_0\beta_0)$ ,  $\eta_\alpha = f^2(j_0\alpha)/f^1(j_0\alpha) \times f^2(j_0\beta_0)$  in two-dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ , we get the coordinate representation (10.4) for the metric function that gives on these manifolds a dimetric physical structure of rank (2,2), as the substitution of it into the initial relation yields an identity.

All the other functional equations (12.2) with the known function  $\Phi$  are solved similarly, though for some of them some difficulties, of purely technical nature, arise of solving them with respect to the variable  $f(i\alpha)$  and of finding a rational way of introducing systems of local coordinates in the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ .

The described method of solution of the functional equation (12.2) may be used with any preassigned function  $\Phi$ . However, unless the equality

$\Phi = 0$  defines a phenomenologically symmetric relation for the physical structure, the coordinate representation obtained of the function  $f(i\alpha)$ , being substituted into the equation (12.2), does not yield an identity with respect to all the points of the cortege  $\langle ijk\dots v, \alpha\beta\gamma\dots\tau \rangle$ . We shall give an interesting example.

While generalizing the phenomenologically symmetric relationship (9.4') for the unimetric physical structure of rank (4,2), it was natural to suppose that the phenomenologically symmetric relationship for the physical structure of rank (5,3) must be written as the equality to zero of the following determinant of 5th order:

$$\begin{vmatrix} 1 & f(i\alpha) & f(i\beta) & f(i\gamma) & f(i\alpha)f(i\beta)f(i\gamma) \\ 1 & f(j\alpha) & f(j\beta) & f(j\gamma) & f(j\alpha)f(j\beta)f(j\gamma) \\ 1 & f(k\alpha) & f(k\beta) & f(k\gamma) & f(k\alpha)f(k\beta)f(k\gamma) \\ 1 & f(l\alpha) & f(l\beta) & f(l\gamma) & f(l\alpha)f(l\beta)f(l\gamma) \\ 1 & f(q\alpha) & f(q\beta) & f(q\gamma) & f(q\alpha)f(q\beta)f(q\gamma) \end{vmatrix} = 0.$$

We shall write that relationship for the cortege  $\langle ij_0k_0l_0q_0, \alpha\beta_0\gamma_0 \rangle$ , solve it with respect to the variable  $f(i\alpha)$ , and introduce coordinates  $x_i, y_i$  in a suitable way in a two-dimensional manifold  $\mathfrak{M}$  and coordinates  $\xi_\alpha, \eta_\alpha, \mu_\alpha, \nu_\alpha$  in a four-dimensional manifold  $\mathfrak{N}$ . As result, we have for the function  $f(i\alpha)$  the following local coordinate representation:

$$f(i\alpha) = (x_i\xi_\alpha + y_i\eta_\alpha + \mu_\alpha)/(x_iy_i + \nu_\alpha).$$

But the substitution of the function obtained into the initial relationship does not turn it into an identity, which fact we established by using the "Maple" computing package. However, that result could have been anticipated in advance, as, according to the classification described at the beginning of §9, no unimetric physical structure of rank (5,3) exists.

The equivalence of the phenomenological and group symmetries described in §8, under which the function  $f$  defining on two sets,  $\mathfrak{M}$  and  $\mathfrak{N}$ , a physical structure is a two-point invariant of some group of their transformations, yields the functional equation (8.6), which is basically different from the equation (12.2) we have discussed, though the solutions of both equations for the function  $f$  must coincide.

In further discussion, it will be suitable to drop in the equation (8.6) the

explicit inclusion of the points  $i$  and  $\alpha$  of the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ , writing it as follows:

$$f(\lambda(x), \sigma(\xi)) = f(x, \xi), \quad (12.11)$$

where  $\lambda : \mathfrak{M} \rightarrow \mathfrak{M}$ ,  $\sigma : \mathfrak{N} \rightarrow \mathfrak{N}$  are locally invertible transformations of the manifolds, and  $x = (x^1, \dots, x^{sm})$ , and  $\xi = (\xi^1, \dots, \xi^{sn})$  are local coordinates in them.

In the general case, the functional equation (12.11), just as the equation (12.2), allows two interpretations. Either both the metric function  $f$  and the functions  $\lambda, \sigma$  defining the transformations of the manifolds  $\mathfrak{M}, \mathfrak{N}$  are unknown. Then, knowing, from Theorem 3 of §8 the dimension of the group of motions of the geometry of two sets defined by the function  $f$ , we perform full, with an accuracy up to equivalence (change of local coordinates), classification of the  $smn$ -dimensional groups of transformations of the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  of the dimensions  $sm$  and  $sn$ , and then via the equation (12.11) come by the nondegenerate two-point invariants. However, for large dimensionalities of the manifolds  $\mathfrak{M}, \mathfrak{N}$  and the group of their transformations, solution of that type of equations is encountered with technical difficulties and can only be done for small dimensionalities. In the latter case, i.e. that when the metric function  $f$  is known, or known are the actions  $\lambda$  and  $\sigma$  of the group of manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ , the functional equation (12.11) may be solved by way of its reduction to a system of differential equations in partial derivatives. Further, we shall discuss some examples of that latter sort.

For the function (9.2):  $f = x + \xi$ , which gives a physical structure of rank (2,2) on one-dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ , the functional equation (12.11):

$$\lambda(x) + \sigma(\xi) = x + \xi$$

has the solution  $\lambda(x) = x + a$ ,  $\sigma(\xi) = \xi - a$  that defines the one-parameter group of motions (9.10) for that function, while the two-point invariant of the group of transformations  $x' = x + a$ ,  $\xi' = \xi - a$  obtained is to be found as the solution of the same functional equation:

$$f(x + a, \xi - a) = f(x, \xi),$$

, and that two-point invariant coincides with the initial metric function with an accuracy up to a scaling transformation:  $f = \chi(x + \xi)$ .

For the function (9.3):  $f = x\xi + \eta$ , which gives a physical structure of rank (3,2) on a one- and a two-dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ , the functional equation (12.11):

$$\lambda(x)\sigma(\xi, \eta) + \rho(\xi, \eta) = x\xi + \eta$$

has the solution  $\lambda(x) = ax + b$ ,  $\sigma(\xi, \eta) = \xi/a$ ,  $\rho(\xi, \eta) = \eta - b\xi/a$ , with  $a \neq 0$ , that defines the one-parameter group of motions (9.11) for that function. The function (9.3) itself may be found as the two-point invariant of that group via the same functional equation (12.11):

$$f(ax + b, \xi/a, \eta - b\xi/a) = f(x, \xi, \eta)$$

with an accuracy up to a scaling transformation:  $f = \chi(x\xi + \eta)$ .

For the function (9.4):  $f = (x\xi + \eta)/(x + \vartheta)$ , which gives a physical structure of rank (4,2) on a one-dimensional and a three-dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ , the functional equation (12.11):

$$\frac{\lambda(x)\sigma(\xi, \eta, \vartheta) + \rho(\xi, \eta, \vartheta)}{\lambda(x) + \tau(\xi, \eta, \vartheta)} = \frac{x\xi + \eta}{x + \vartheta}$$

has the solution that defines the three-parameter group of motions (9.12). As to the metric function (9.4) itself, it may be found as the two-point invariant of that group via the functional equation (12.11):

$$f\left(\frac{ax + b}{cx + d}, \frac{d\xi - c\eta}{d - c\vartheta}, \frac{a\eta - b\xi}{d - c\vartheta}, \frac{a\vartheta - b}{d - c\vartheta}\right) = f(x, \xi, \eta, \vartheta).$$

with an accuracy up to a scaling transformation  $\chi(f) \rightarrow f$ .

For the two-component function (10.3):  $f^1 = x + \xi$ ,  $f^2 = y + \eta$ , which gives a dimetric physical structure of rank (2,2) on two-dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ , the functional equation (12.11):

$$\left. \begin{aligned} \lambda^1(x, y) + \sigma^1(\xi, \eta) &= x + \xi, \\ \lambda^2(x, y) + \sigma^2(\xi, \eta) &= y + \eta \end{aligned} \right\}$$

has the solution  $\lambda^1(x, y) = x + a$ ,  $\lambda^2(x, y) = y + b$ ,  $\sigma^1(\xi, \eta) = \xi - a$ ,  $\sigma^2(\xi, \eta) = \eta - b$  that defines a two-parameter group of motions for that function:  $x' = x + a$ ,  $y' = y + b$ ,  $\xi' = \xi - a$ ,  $\eta' = \eta - b$ . The function (10.3) is found as the two-point invariant of that group via the functional equation (12.11):

$$f(x + a, y + b, \xi - a, \eta - b) = f(x, y, \xi, \eta).$$

with an accuracy up to a two-dimensional scaling transformation:  $f^1 = \chi^1(x + \xi, y + \eta)$ ,  $f^2 = \chi^2(x + \xi, y + \eta)$ .

For the second function (10.4):  $f^1 = (x + \xi)y$ ,  $f^2 = (x + \xi)\eta$ , which gives another physical structure of rank (2,2), the functional equation (12.11):

$$\left. \begin{aligned} (\lambda^1(x, y) + \sigma^1(\xi, \eta))\lambda^2(x, y) &= (x + \xi)y, \\ (\lambda^1(x, y) + \sigma^1(\xi, \eta))\sigma^2(\xi, \eta) &= (x + \xi)\eta \end{aligned} \right\}$$

has the solution  $\lambda^1(x, y) = ax + b$ ,  $\lambda^2(x, y) = y/a$ ,  $\sigma^1(\xi, \eta) = a\xi - b$ ,  $\sigma^2(\xi, \eta) = \eta/a$ , with  $a \neq 0$ , that defines a two-parameter group of motions for that function:  $x' = ax + b$ ,  $y' = y/a$ ,  $\xi' = a\xi - b$ ,  $\eta' = \eta/a$ . The metric function itself is come by as the two-point invariant of the group of motions via solving the functional equation (12.11):

$$f(ax + b, y/a, a\xi - b, \eta/a) = f(x, y, \xi, \eta)$$

with an accuracy up to a scaling transformation  $\chi^1(f^1, f^2) \rightarrow f^1$ ,  $\chi^2(f^1, f^2) \rightarrow f^2$ .

For all the other functions, (10.5) to (10.12), that give dimetric physical structures of ranks (3,2), (4,2) and (5,2), as well as for the functions (10.18)–(10.28), that give trimetric physical structures of rank (2,2), the functional equation (12.11) is considered similarly.

Functional equations appear quite naturally in the theory of groups of transformations too, for that theory is inherently related with that of physical structures. For the groups of transformations  $G^r(\lambda)$  and  $H^r(\sigma)$  of the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  with the actions  $x' = \lambda(x, a)$  and  $\xi' = \sigma(\xi, \alpha)$ , where  $a \in G^r$  and  $\alpha \in H^r$ , and the rules of multiplication in the parameter groups  $G^r$  and  $H^r$  being  $ab = \varphi(a, b)$  and  $\alpha\beta = \psi(\alpha, \beta)$  respectively, their isomorphism is established by the solution of the functional equation

$$u(\varphi(a, b)) = \psi(u(a), u(b)) \quad (12.12)$$

with respect to the biunivocal mapping  $u : G^r \rightarrow H^r$ , while their similarity is established by the solution of the system of two functional equations: (12.12) and

$$v(\lambda(x, a)) = \sigma(v(x), u(a)) \quad (12.13)$$

with respect to the invertible mappings  $u : G^r \rightarrow H^r$  and  $v : \mathfrak{M} \rightarrow \mathfrak{N}$ .



It is evident that similarity is only possible if there is coincidence of the dimensions of the manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$ .

The weak equivalence of the groups of transformations  $G^r(\lambda)$  and  $G^r(\sigma)$ , which have one and the same parameter group  $G^r$ , with the actions  $x' = \lambda(x, a)$  and  $\xi' = \sigma(\xi, a)$  is established by the solution of the system of the functional equations (12.12) and (12.13), where  $\psi = \varphi$ , with respect to the automorphism  $u : G^r \rightarrow G^r$  and the invertible mapping  $v : \mathfrak{M} \rightarrow \mathfrak{N}$ , and their strong equivalence is established by the solution of the functional equation

$$w(\lambda(x, a)) = \sigma(w(x), a) \quad (12.14)$$

with respect to the invertible mapping  $w : \mathfrak{M} \rightarrow \mathfrak{N}$ .

We shall note that a case is quite possible when the system of functional equations (12.12) and (12.13) with  $\psi = \varphi$  *does* have a solution, while the functional equation (12.14) has no solution, i.e. the groups of transformations  $G^r(\lambda)$  and  $G^r(\sigma)$  while equivalent weakly have at the same time no strong equivalence.

### §13. Interpretations of physical structures

Physical structures, as mathematical forms, may have various meanings, i.e. they may have various physical and geometric interpretations. Now we shall give examples.

We shall write Newton's 2nd law:  $F = ma$ , considered in the Introduction (see the equations (B.17) and (B.18)), in the multiplicative canonical form:

$$f = x\xi, \quad f(i\alpha)f(j\beta) - f(i\beta)f(j\alpha) = 0, \quad (13.1)$$

where, for example,  $f(i\alpha) = x_i\xi_\alpha$ , and introduce a single notation for the functions and the coordinates:

$$f = a, \quad x = 1/m, \quad \xi = F. \quad (13.2)$$

The canonical equations (13.1) are purely mathematical relationships that can be filled with different physical contents. For Newton's 2nd law,

according to the designations (13.2), the function  $f$  is the acceleration  $a$  of a body under the impact of the accelerator, the coordinate  $x$  gives the value that is the inverse of the mass  $m$  of the body, and the coordinate  $\xi$  coincides with the force  $F$  of the accelerator.

The justification of such an approach to the canonical form (13.1) is in the rich opportunities it presents of different physical interpretations, i.e. it is not only the Newton's Second Law of Mechanics that can be reduced to it, but many other laws of physics.

For example, let us consider the optical Law of Refraction, for the case of a beam of light falling from the vacuum into some refracting medium, whose formula is known to be  $\sin \varphi / \sin \psi = n$ , that reads:

*The ratio of the sine of the angle of incidence to the sine of the angle of refraction is equal to the index of refraction of the medium.*

It is easy to see that the Law of Refraction, just as Newton's Second Law, relates physical quantities of different natures. Indeed, the angle of incidence  $\varphi$  only characterizes the beam, while the refractive index  $n$  characterizes the medium. But the angle of the refraction  $\psi$ , directly measured by experiment, characterizes simultaneously the beam and the optical medium, establishing whereby their interaction.

To stress that circumstance, we shall introduce a set of incident beams  $\mathfrak{M} = \{i, j, k, \dots\}$  and a set of optical mediums  $\mathfrak{N} = \{\alpha, \beta, \gamma, \dots\}$ . Then, for an arbitrary beam  $i \in \mathfrak{M}$  with the incidence angle  $\varphi_i$  and an arbitrary medium  $\alpha \in \mathfrak{N}$  with the refractive index  $n_\alpha$ , the formula of the law of refraction is as follows:

$$\sin \varphi_i / \sin \psi_{i\alpha} = n_\alpha, \quad (13.3)$$

where we can see that the *mathematical* natures of the quantities  $\varphi$ ,  $n$  and  $\psi$  are different too, as the two former quantities are one-index ones and characterize the incident beam and the optical medium, while the third is a two-index one and characterizes the interaction of the incident beam and the medium.

The critical part in the law of refraction (13.3) is obviously that played by the angle of refraction, and so it is natural to rewrite that law in the phenomenologically symmetric form that should only contain the angles of

refraction measured by experiment. To do it it is necessary, just as in the case of Newton's Second Law, to take two elements from each set, that is two beams  $i, j$  of the set of incident beams  $\mathfrak{M}$  and two mediums  $\alpha, \beta$  of the set of optical mediums  $\mathfrak{N}$ . The relation among the four possible angles of refraction  $\psi_{i\alpha}, \psi_{i\beta}, \psi_{j\alpha}, \psi_{j\beta}$  is easily found by using the formula (13.3):

$$\sin \psi_{i\alpha} \sin \psi_{j\beta} - \sin \psi_{i\beta} \sin \psi_{j\alpha} = 0, \quad (13.3')$$

which equation gives the Law of Refraction in the phenomenologically symmetric form. We shall note that the equations (13.3) and (13.3') of the law of refraction are reduced to the multiplicative canonical form (13.1) if we set

$$f = \sin \psi, \quad x = \sin \varphi, \quad \xi = 1/n. \quad (13.4)$$

Thus, the canonical form (13.1) may be filled with different physical contents if we point out precisely what kind of physical objects the sets  $\mathfrak{M}$  and  $\mathfrak{N}$  comprise, and what measurement procedure assigns two objects of these sets the number that characterizes their interaction. We call a mathematical object for which the equations (13.1) are satisfied a physical structure, for it has, as was demonstrated above, various physical interpretations. It is also said (see §8) that the function  $f$  defines on the sets  $\mathfrak{M}$  and  $\mathfrak{N}$  a *physical structure of rank (2,2)* because the second equation (13.1) defines the functional relation of the values of that function for any two elements  $i, j$  of the former set and any two elements  $\alpha, \beta$  of the latter.

Ohm's law for a closed circuit:  $I = \mathcal{E}/(R + r)$  was discussed in detail in the Introduction (see the expressions (B.19) and (B.20)). Let us endow it with the canonical form, introducing suitable designation as follows:  $R = x$ ,  $1/\mathcal{E} = \xi$ ,  $r/\mathcal{E} = \eta$ ,  $1/I = f$ :

$$\left. \begin{aligned} f &= x\xi + \eta, \\ \begin{vmatrix} f(i\alpha) & f(i\beta) & 1 \\ f(j\alpha) & f(j\beta) & 1 \\ f(k\alpha) & f(k\beta) & 1 \end{vmatrix} &= 0, \end{aligned} \right\} \quad (13.5)$$

where, for example,  $f(i\alpha) = x_i \xi_\alpha + \eta_\alpha$ .

It appears that the canonical form (13.5) may have still another physical meaning. Now we shall consider the law of the linear thermal expansion

of solid bodies:  $L = L_0(1 + Et)$ , where  $L$  is the length of a bar at the given temperature  $t$ , in degrees of centigrade,  $L_0$  is its length at the zero temperature, and  $E$  is the thermal expansion coefficient. In that law, as in Ohm's Law, physically different quantities are related one with another. Indeed, the temperature  $t$  characterizes the thermostat where the measuring of the length of a bar is performed, while the original length  $L_0$  and the thermal expansion coefficient  $E$  characterize the bar. The length  $L$  depends on both the bar and the thermostat where the bar is placed.

We shall stress that difference, introducing a set of thermostats  $\mathfrak{M} = \{i, j, k, \dots\}$  and a set of bars  $\mathfrak{N} = \{\alpha, \beta, \gamma, \dots\}$ . The thermostat  $i$  is characterized by its temperature  $t_i$  of it measured by a thermometer, and the bar  $\alpha$  is characterized with its original length  $L_{0\alpha}$  at the zero temperature and the coefficient of volume expansion  $E_\alpha$ , which is, in the linear approximation, considered to be constant, the length  $L_{i\alpha}$  of the bar  $\alpha$  placed into the thermostat  $i$  measured by experiment being a two-index quantity. Then, the law of thermal expansion acquires the form as follows:

$$L_{i\alpha} = L_{0\alpha}(1 + E_\alpha t_i), \quad (13.6)$$

which demonstrates clearly the physical and mathematical heterogeneity of the quantities it includes.

The law of thermal expansion (13.6) and Ohm's law may be written in the single canonical form (13.5), if we introduce the designation as follows:  $t = x$ ,  $EL_0 = \xi$ ,  $L_0 = \eta$ ,  $L = f$ . Then, the phenomenologically symmetric form of either of the laws will, obviously, be the functional relation defined by the second equation in (13.5).

The canonical form (13.5), single for two different laws of physics, may be disengaged from any physical meaning and considered as a sheerly mathematical object that we name, due to its origin, a *physical structure of rank (3,2)* which is also a phenomenologically symmetric geometry of two sets of the same rank, for the metric function  $f = x\xi + \eta$  is a two-point one and its value  $f(i\alpha)$  may, in some generalized sense, be termed as the distance between a point  $i$  and a point  $\alpha$  of different sets,  $\mathfrak{M}$  and  $\mathfrak{N}$ .

Let us dwell more on interpretations of the *physical structure of rank*

(4,2), whose canonical form was given in §9 (see (9.4) and (9.4')):

$$\left. \begin{aligned} f &= (x\xi + \eta)/(x + \vartheta), \\ \left| \begin{array}{cccc} f(i\alpha) & f(i\beta) & f(i\alpha)f(i\beta) & 1 \\ f(j\alpha) & f(j\beta) & f(j\alpha)f(j\beta) & 1 \\ f(k\alpha) & f(k\beta) & f(k\alpha)f(k\beta) & 1 \\ f(l\alpha) & f(l\beta) & f(l\alpha)f(l\beta) & 1 \end{array} \right| &= 0, \end{aligned} \right\} \quad (13.7)$$

where, for example,  $f(i\alpha) = (x_i\xi_\alpha + \eta_\alpha)/(x_i + \vartheta_\alpha)$ . The validity of the equation that expresses the phenomenological symmetry of that structure may be assured by way of direct substitution into it the metric function, using the method of expansion of the determinant with respect to the sum in the column. Or, to make the whole business simpler, using computing packages "Maple" and "Mathematica" that can compute determinants and ranks of matrices. First, we shall take the optics of the thick lens (see [1], pp. 506-508). Its formula looks quite similar to that of the thin lens:

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{F},$$

in which  $a$  is the distance from the object to the centre of the lens along the principal axis,  $b$  is the corresponding distance for the image and  $F$  is the focal distance of the lens. For the case of the thick lens the values  $a$  and  $b$  are of some different meaning. The thing is, they are measured along the principal axis too, but not to the centre of the lens, but to the two principal planes of it. Suppose  $x$  is the distance along the principal axis from the object to the nearest point on the lens surface, and  $\lambda$  is that from the lens to the nearest principal plane. Then  $a = x + \lambda$ . Similarly, suppose  $u$  is the distance from the image to a point on the other surface of the lens, and  $\sigma$  is that from the lens to the nearest principal plane. For the sake of simplicity, it is suitable to draw a biconvex thick lens with  $F > 0$ , because all the values in the formula will be positive. We shall substitute the expressions for the values  $a$  and  $b$  that we have mentioned into the formula of the thick lens and solve it with respect to the distance from the lens to the image:

$$u = \frac{x(F - \sigma) + (\lambda + \sigma)F - \lambda\sigma}{x + \lambda - F}. \quad (13.8)$$

Now we shall consider a set of objects  $\mathfrak{M}$  and a set of thick lenses  $\mathfrak{N}$  that are used to build their images. The former of the sets is a one-dimensional manifold whose points are defined by the coordinate  $x$ , while the latter is a three-dimensional manifold, and the points of it are defined by the coordinates  $F, \lambda, \sigma$ . In the law (13.8), derived from the formula of the thick lens, it is quantities of different nature that are related with one another. The coordinate  $x$  characterizes the object, the coordinates  $F, \lambda, \sigma$  characterize the lens, and the value  $u$  characterizes the "interaction" of the object and the lens. In order to throw that into relief, we shall substitute into the law (13.8) some an object  $i \in \mathfrak{M}$  and a lens  $\alpha \in \mathfrak{N}$ :

$$u_{i\alpha} = \frac{x_i(F_\alpha - \sigma_\alpha) + (\lambda_\alpha + \sigma_\alpha)F_\alpha - \lambda_\alpha\sigma_\alpha}{x_i + \lambda_\alpha - F_\alpha}.$$

The phenomenological symmetry of the law (13.8) is revealed forthwith if we reduce it to the canonical form (13.7) by way of obvious changes of coordinates as follows:  $x \rightarrow x$ ,  $F - \sigma \rightarrow \xi$ ,  $(\lambda + \sigma)F - \lambda\sigma \rightarrow \eta$ ,  $\lambda - F \rightarrow \vartheta$  together with change of designation of the quantity measured:  $u \rightarrow f$ . Then, the phenomenologically symmetric form of the law (13.8) for the thick lens will be the equation from (13.7).

A geometric interpretation of the physical structure of rank (4,2) is built as follows (see [1], pp. 501-502). Suppose  $\mathfrak{M}$  is a one-parameter set of straight lines on the Euclidean plane passing through the coordinate origin. Every such straight line is uniquely determined by the angle  $\varphi$  between the line itself and the abscissa axis, and  $-\pi/2 < \varphi \leq +\pi/2$ . Suppose, further, the second set is a three-parameter set  $\mathfrak{N}$  of straight lines passing through the points  $(a, b)$  at different angles  $\theta$  to the abscissa axis, and again  $-\pi/2 < \theta \leq +\pi/2$ . We shall assign two straight lines from these sets the quantity defined by the expression

$$f = \frac{-\frac{a}{\cos \theta} \operatorname{tg} \varphi + \frac{b}{\cos \theta}}{\operatorname{tg} \varphi - \operatorname{tg} \theta},$$

whose modulus is equal to the distance from the point of their intersection to the point  $(a, b)$ . Introducing change of coordinates:  $\operatorname{tg} \varphi \rightarrow x$ ,  $-a/\cos \theta \rightarrow \xi$ ,  $b/\cos \theta \rightarrow \eta$ ,  $-\operatorname{tg} \theta \rightarrow \vartheta$ , we arrive at the canonical form of the metric function (13.7) that gives on a one-dimensional and a three-dimensional

manifolds a phenomenologically symmetric geometry of two sets (a physical structure) of rank  $(4,2)$ .

#### §14. The unresolved problems in the theory of physical structures

This paragraph is written with an object of giving a brief overview of the mathematical problems of the theory of physical structures, in the hope that some of them may be interesting to the reader.

The phenomenological symmetry of a physical structure is, under Theorem 3 of §8, equivalent to the group symmetry of it in the sense as follows: a nondegenerate  $s$ -component metric function  $f$  allows an  $smn$ -dimensional group of motions if, and only if, it defines on an  $sm$ - and an  $sn$ -dimensional manifolds a physical structure of rank  $(n+1, m+1)$ . And that means that the task of solving the functional equation (12.2), in which the unknowns are the metric function  $f$  and the function  $\Phi$ , is equivalent to the task of solving the functional equation (12.11), in which the unknowns are the same metric function  $f$  and the actions  $\lambda$ , and  $\sigma$  of the Lie group in the manifolds.

We shall note that the methods of solving the equation (12.2) and those for the equation (12.11) are quite different. But with respect to the principal task that is being solved i.e. that of classifying physical structures, those methods are mutually complementary, making the result of the effort at classification more reliable. Some metric functions are found only as solutions of the functional equation (12.2), others only as solutions of the functional equation (12.11). There are still others, whose solutions were found by way of solving both of the equations. The complete classification of the polymetric physical structures, however, save for the monometric ones (see §9), has not been built yet. So it seems it makes sense to give a brief overview of all problems of classification of the physical structures on two manifolds (see the table below). Trying to solve them makes sense not only mathematically, but in the physical sense too, as the result may be possible forms of fundamental laws of physics. The author hopes that some readers will succeed in not only combining the already known methods, but in finding such new ones

that would make it possible to carry on and complete the work of classifying the polymetric physical structures of arbitrary rank.

A classification of polymetric physical structures									
Nº	$s$	$m$	$n$	$sm$	$sn$	$(n + 1, m + 1)$	$smn$	solved	source
1	1	$\geq 1$	$\geq m$	$m$	$n$	$(n + 1, m + 1)$	$mn$	+	§9
2	2	1	$\geq 1$	1	$2n$	$(n + 1, 2)$	$2n$	+	§10
3	2	$\geq 2$	$\geq m$	$2m$	$2n$	$(n + 1, m + 1)$	$2mn$	–	–
4	3	1	1	3	3	(2,2)	3	+	§10
5	3	1	$\geq 2$	3	$3n$	$(n + 1, 2)$	$3n$	–	–
6	3	$\geq 2$	$\geq m$	$3m$	$3n$	$(n + 1, m + 1)$	$3mn$	–	–
7	4	1	1	4	4	(2,2)	4	+	§10
8	4	1	$\geq 2$	4	$4n$	$(n + 1, 2)$	$4n$	–	–
9	4	$\geq 2$	$\geq m$	$4m$	$4n$	$(n + 1, m + 1)$	$4mn$	–	–
10	$\geq 5$	$\geq 1$	$\geq m$	$\geq 5m$	$\geq 5n$	$(n + 1, m + 1)$	$\geq 5mn$	–	–

We remind that  $s \geq 1$  is the number of the components of the nondegenerate metric function  $f = (f^1, \dots, f^s)$  that gives on an  $sm$ -dimensional and an  $sn$ -dimensional manifolds  $\mathfrak{M}$  and  $\mathfrak{N}$  a physical structure (a phenomenologically symmetric geometry of two sets) of rank  $(n + 1, m + 1)$  endowed with a group symmetry of degree  $smn$ . The condition  $n \geq m$  is introduced with the purpose of decreasing the number of lines in the table, because the classification result is symmetrical with respect to the permutation of the natural numbers  $m$  and  $n$ . In the last column but one of the table the 'plus' and 'minus' signs are to inform whether the problem has been solved or has not been solved respectively. The last column of the table gives the number of the paragraph of this monograph where the classification has been given and the methods used in building are described it, or the names of works are given where they are treated in greater detail.



## CONCLUSION

Thus, it has been established that binary phenomenologically symmetric geometries of one and two sets are endowed with a group symmetry, and, besides, are pregnant with essential physical and mathematical meaning. So, the principal task for the theory of physical structures (TPS) is that of complete classification of them. That work is far from being complete, which gives an inquisitive mathematician chance to try oneself in a field of scientific investigation new to them.

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## A heap and the physical structure of rank (2,2)

by A.N. Borodin

As is well known [1], a *heap* is an algebra  $G$  with a ternary operation  $\varphi : G^3 \rightarrow G$  satisfying the following identities:

$$\varphi(\varphi(x, y, z), u, v) = \varphi(x, \varphi(u, z, y), v) = \varphi(x, y, \varphi(z, u, v)), \quad (1)$$

$$\varphi(x, y, y) = \varphi(y, y, x) = x. \quad (2)$$

In the medium term of the identities (1), there is a quite ambiguous permutation present of the elements  $y$  and  $u$  of the cortege  $\langle xyzuv \rangle$ . It turns out that that medium term in the definition of a heap may be altogether omitted.

**Lemma 1.** *The identities (1) and (2), if satisfied with a ternary operation  $\varphi$ , are equivalent to the identities*

$$\varphi(\varphi(x, y, z), u, v) = \varphi(x, y, \varphi(z, u, v)), \quad (3)$$

$$\varphi(x, y, y) = \varphi(y, y, x) = x. \quad (4)$$

It follows from Lemma 1 that the definition of a heap by the identities (1), (2), as in the lectures of A.G. Kurosh [1], is equivalent to its definition by the identities (3), (4).

**Definition 1.** *An algebra  $G$  with a ternary operation  $\varphi$  is a heap, if that operation satisfies the identities (3), (4).*

**Lemma 2.** *The identities (3), (4), satisfied with the ternary operation  $\varphi$ , are equivalent to the identities*

$$\varphi(x, y, z) = \varphi(\varphi(x, y, s), s, z) = \varphi(x, s, \varphi(s, y, z)), \quad (5)$$

$$\varphi(x, y, y) = \varphi(y, y, x) = x. \quad (6)$$

Detailed proof of Lemma 2 as well as that of Lemma 1 may be found in the author's note [2].

**Definition 2.** *An algebra  $G$  with a ternary operation  $\varphi$  is a heap, if that operation satisfies the identities (5), (6).*

**Lemma 3.** *The definition 1 and the definition 2 of a heap as an algebra  $G$  with a ternary operation  $\varphi$  satisfying the identities (3), (4) and the identities (5), (6) respectively are equivalent.*

To define a heap with the identities (5) and (6) seems more natural, as they are the corollary of the principle of phenomenological symmetry of the theory of physical structures [3].

Let there be three sets –  $\mathfrak{M}, \mathfrak{N}$  and  $G$  – of arbitrary nature, as well as a function  $f : \mathfrak{M} \times \mathfrak{N} \rightarrow G$  that assigns to each pair  $\langle i\alpha \rangle$  from the direct product  $\mathfrak{M} \times \mathfrak{N}$  some element  $f(i\alpha)$  of the set  $G$ . With respect to the function  $f$ , we shall assume that the condition is satisfied as follows:

**A.** *For any elements  $\beta \in \mathfrak{N}$  and  $j \in \mathfrak{M}$  the mappings  $\mathfrak{M} \times \{\beta\} \rightarrow G$  and  $\{j\} \times \mathfrak{N} \rightarrow G$  are surjective.*

We shall introduce still one more function –  $F : \mathfrak{M}^2 \times \mathfrak{N}^2 \rightarrow G^4$ , – by assigning to the cortege  $\langle ij, \alpha\beta \rangle \in \mathfrak{M}^2 \times \mathfrak{N}^2$  a point  $\langle f(i\alpha), f(i\beta), f(j\alpha), f(j\beta) \rangle \in G^4$ , whose coordinates in  $G^4$  are the images of the corresponding pairs ordered with respect to the original cortege.

**Definition 3.** *We shall say that the function  $f : \mathfrak{M} \times \mathfrak{N} \rightarrow G$  that satisfies the condition A gives on the sets  $\mathfrak{M}$  and  $\mathfrak{N}$  a physical structure of rank (2, 2), if there exists such a ternary algebraic operation  $\varphi : G^3 \rightarrow G$  for which the relation is satisfied as follows:*

$$f(i\alpha) = \varphi(f(i\beta), f(j\beta), f(j\alpha)). \quad (7)$$

The relation (7), true for any cortege  $\langle ij, \alpha\beta \rangle \in \mathfrak{M}^2 \times \mathfrak{N}^2$ , expresses the essence of the *principle of phenomenological symmetry* of the theory of physical structures. Since it is a functional equation it imposes on the original function  $f$  a strong enough restriction.

**Theorem 1.** *The ternary algebraic operation  $\varphi$  from the Definition 3 of*

the physical structure of rank  $(2, 2)$  that establishes the phenomenologically symmetric relation (7) defines a heap on the set  $G$ .

We shall set in the relation (7)  $i = j : f(i\alpha) = \varphi(f(i\beta), f(i\beta), f(i\alpha))$  and  $\alpha = \beta : f(i\alpha) = \varphi(f(i\alpha), f(j\alpha), f(j\alpha))$ . Under the condition **A**, the pairs of the variables  $f(i\alpha), f(i\beta)$  and  $f(i\alpha), f(j\alpha)$  are independent. Introducing for them the designation  $x = f(i\alpha)$ ,  $y = f(i\beta)$ , for the former case, and  $x = f(i\alpha)$ ,  $y = f(j\alpha)$ , for the latter, yields the identities (6). We shall take an element  $k$  of the set  $\mathfrak{M}$  and write the relation (7) for the corteges  $\langle ik, \alpha\beta \rangle$  and  $\langle jk, \alpha\beta \rangle$ :

$$\left. \begin{aligned} f(i\alpha) &= \varphi(f(i\beta), f(k\beta), f(k\alpha)), \\ f(j\alpha) &= \varphi(f(j\beta), f(k\beta), f(k\alpha)). \end{aligned} \right\} \quad (7')$$

Out of the three relations (7), (7'), we easily establish the equality

$$\varphi(f(i\beta), f(k\beta), f(k\alpha)) = \varphi(f(i\beta), f(j\beta), \varphi(f(j\beta), f(k\beta), f(k\alpha))),$$

with independent, under the condition **A**, variables  $f(i\beta), f(k\beta), f(k\alpha), f(j\beta)$ . Introducing the designation  $x = f(i\beta)$ ,  $y = f(k\beta)$ ,  $z = f(k\alpha)$ ,  $s = f(j\beta)$  for them yields one of the identities (5). Let us take, further, an element  $\gamma$  of the set  $\mathfrak{M}$  and write the relation (7) for the corteges  $\langle ij, \alpha\gamma \rangle$  and  $\langle ij, \beta\gamma \rangle$ :

$$\left. \begin{aligned} f(i\alpha) &= \varphi(f(i\gamma), f(j\gamma), f(j\alpha)), \\ f(i\beta) &= \varphi(f(i\gamma), f(j\gamma), f(j\beta)). \end{aligned} \right\} \quad (7'')$$

From the three relations (7), (7'') there follows the equality

$$\varphi(f(i\gamma), f(j\gamma), f(j\alpha)) = \varphi(\varphi(f(i\gamma), f(j\gamma), f(j\beta)), f(j\beta), f(j\alpha)),$$

with the independent, under the condition **A**, variables  $f(i\gamma), f(j\gamma), f(j\alpha), f(j\beta)$ . Introducing the designation  $x = f(i\gamma)$ ,  $y = f(j\gamma)$ ,  $z = f(j\alpha)$ ,  $s = f(j\beta)$  for them yields the other of the identities (5). Thus, the identities (5) and (6), that are part of the Definition 2 of a heap, are established, which makes the proof of Theorem 1 complete.

The difference of the roles played by the identities (5) and (6) is worth noting too. The former are obviously basic ones, such that look like some functional relations that define a heap, while the latter only reflect its



minor characteristics. So it makes sense in a new definition of a heap to retain the identities (5), and substitute for the identities (6) some more natural condition imposed on the ternary operation  $\varphi$ . Such condition may be obtained from the same phenomenologically symmetric relation (7) for the physical structure of rank (2,2).

**Lemma 4.** *The ternary algebraic operation  $\varphi$  from the definition 3 of the physical structure of rank (2,2) satisfies the following necessary condition:*

**B.** *For any two elements  $q, h \in G$ , the mappings  $x \mapsto \varphi(x, q, h)$ ,  $x \mapsto \varphi(q, x, h)$ ,  $x \mapsto \varphi(q, h, x)$  are surjective.*

Let us consider the first of the mappings,  $x \mapsto \varphi(x, q, h)$ . Under the condition **A** there exists such a pair  $\langle j\alpha \rangle$  for which  $f(j\alpha) = h$ . Next, according to the same condition **A**, for the points  $j \in \mathfrak{M}$  and  $\alpha \in \mathfrak{N}$  of the previous pair there exist points  $i \in \mathfrak{M}$  and  $\beta \in \mathfrak{N}$ , for which  $f(j\beta) = q$  and  $f(i\alpha) = p$ , where  $p$  is an arbitrary element from  $G$ . But then, setting  $x = f(i\beta)$  yields, with respect to the relation (7),  $p = \varphi(x, q, h)$ . I. e. an arbitrary element  $p \in G$  with the mapping  $x \mapsto \varphi(x, q, h)$  has at least one preimage, which proves the mapping being surjective. The surjectivity of the mappings  $x \mapsto \varphi(q, x, h)$  and  $x \mapsto \varphi(q, h, x)$  is established in absolutely the same way. Lemma 4 has been proved.

The condition **B** has an equivalent form, which algebraists have been more accustomed to:

**B'.** *For any three elements  $p, q, h$  of the set  $G$  each of the equations  $p = \varphi(x, q, h)$ ,  $p = \varphi(q, x, h)$  and  $p = \varphi(q, h, x)$  has a solution with respect to  $x$ .*

We shall replace the identities (6), that do not look quite natural, in the definition (2) of a heap with a more natural condition **B**, which is also a corollary of the phenomenological symmetry.

**Definition 4.** The algebra  $G$  with a ternary operation  $\varphi$  satisfying the condition **B** (or the condition **B'** that is equivalent to it) is a heap if it

satisfies the two identities as follows:

$$\left. \begin{aligned} \varphi(x, y, z) &= \varphi(\varphi(x, y, s), s, z), \\ \varphi(x, y, z) &= \varphi(x, s, \varphi(s, y, z)). \end{aligned} \right\} \quad (8)$$

**Lemma 5.** *Definition 2 and definition 4 of a heap as of an algebra  $G$  with a ternary operation  $\varphi$  satisfying the four identities (5) and (6) or, under condition **B**, the two identities (8) respectively, are equivalent.*

The identities (5) and (8) coincide, so let us first get the identities (6) from the condition **B** and the identities (8). We shall write the first and the second of the identities (8) for the corteges  $\langle xyxy \rangle$  and  $\langle yxyx \rangle$  respectively:  $\varphi(x, y, y) = \varphi(\varphi(x, y, y), y, y)$ ,  $\varphi(y, y, x) = \varphi(y, y, \varphi(y, y, x))$ . Under the condition **B**, the mappings  $x \mapsto \varphi(x, y, y)$  and  $x \mapsto \varphi(y, y, x)$  are surjective ones. Introducing corresponding redesignation of the elements  $\varphi(x, y, y)$  and  $\varphi(y, y, x)$  from  $G$  yields the identities (6). Now we shall demonstrate that the condition **B** is a corollary of the identities (5),(6). We shall assume the contrary, i.e. that there exist three elements  $p, q, h$  from the set  $G$ , such that one of the three equations of the condition **B'** has no solution. Without loss of generality, it is possible to assume that it is the equation  $p = \varphi(x, q, h)$  that has no solution. We shall write the former of the identities (5) for the cortege  $\langle phhq \rangle$ :  $\varphi(p, h, h) = \varphi(\varphi(p, h, q), q, h)$ , whereof, by using one of the identities (6), we get:  $p = \varphi(\varphi(p, h, q), q, h)$ . Thus, the equation  $p = \varphi(x, q, h)$  does have a solution  $x = \varphi(p, h, q)$ , which is in contradiction with the above assumption. The two other equations from the condition **B'** are investigated similarly. The contradictions established in the process demonstrate that the condition **B'** (or the condition **B** equivalent to it) is a corollary of the identities (5),(6). Lemma 5 has been proved.

The condition **B** may seem too strong, and the more so because in the proof of Lemma 5, in obtaining the identities (6), the condition **B** was not used in corpore, as it was only the surjectivity of the mappings  $x \mapsto \varphi(x, y, y)$  and  $x \mapsto \varphi(y, y, x)$  for an arbitrary element  $y$  that mattered. We shall define that weaker condition:

**C.** For any element  $q \in G$  the mappings defined by the functions  $x \mapsto \varphi(x, q, q)$  and  $x \mapsto \varphi(q, q, x)$  are surjective.

An equivalent to it is the variant as follows:

**C'.** For any two elements  $p, q \in G$ , each of the equations  $p = \varphi(x, q, q)$  and  $p = \varphi(q, q, x)$  has a solution with respect to  $x \in G$ .

We shall note, however, that the weak condition **C** seems less natural, than the stronger condition **B**.

**Definition 5.** The algebra  $G$  with a ternary operation  $\varphi$ , which satisfies the condition **C** (or the equivalent condition **C'**) and the identities (8), is a heap.

**Lemma 6.** Definition 2 and Definition 5 of a heap as an algebra  $G$  with a ternary operation  $\varphi$  satisfying the four identities (5), (6) or, under condition **C**, the two identities (8) are equivalent.

The proof of Lemma 6, in the first part of it, that where the identities (6) are obtained, repeats the respective part of the proof of Lemma 5, and the surjectivity of the mappings  $x \mapsto \varphi(x, q, q)$  and  $x \mapsto \varphi(q, q, x)$ , stipulated under the condition **C**, is a direct corollary of the identities (6).

**Theorem 2.** All the four definitions of a heap, i.e. Definitions 1, 2, 4, 5 are equivalent one with another.

Theorem 2 is a congregate corollary of Lemmas 3, 5, 6, which establish the transitive equivalence of the pairs of definitions 1 and 2, 2 and 4, and 2 and 5.

The author expresses gratitude to Professor G.G. Mihailichenko and the participants of the academic Theory of physical structures seminar of FMF GAGU for the support of that research and the discussion of the results obtained.

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**Gennady Grigorievich Mihailichenko**

was born in 1942. In 1967, he graduated from the Department of Physics of the Novosibirsk university, and in 1970 finished the post-graduate school at the same University. Up to 1994, he was an associate professor at the Chair of Theoretical Physics of the Novosibirsk Pedagogical university, since 1994 has been a professor at the Chair of Physics of Gorno-Altai University. A Doctor (1994, a post-Ph.D. degree under the Russian system) of Physical and Mathematical Sciences and a Professor (2000).

G.G. Mihailichenko is the author of more than sixty notes and of five monographs. This monograph: *The Mathematical Basics and the Results of the Theory of physical structures*, which is his sixth monograph, is a sort of a conclusion of the author's academic activity.

**G.G. Mikhailichenko**

Professor of Chair of Physics.

Gorno-Altai State University.

1 Lenkin str. Gorno-Altai, 649000, Russia.

e-mail: mikhailichenko@gasu.ru

Academic edition

**Gennady Grigorievich Mihailichenko**

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AND RESULTS OF THE  
THEORY OF PHYSICAL STRUCTURES**

Gorno-Altai State university press  
649000, 1 Lenkina Street, Gorno-Altai, Russian Federation

Sent to the press: 05.06.2012 Format:  $60 \times 84/16$ .  
Duplication paper (paper for copy machines). "Riso" type  
9,12 sheets. 60 copies printed.  
Order № 83.

Printed by the Printing department of  
Gorno-Altai State university  
649000, 1 Lenkina Street, Gorno-Altai, Russian Federation  
Order No. 83.